$$\Big[(a^{m-1}-1)-\sum_i (a^{(m/p_i-1)}-1)$$
$$+\sum_{ij}(a^{(m/p_ip_j-1)}-1)-...$$
$$...+(-1)^k(a^{(m/p_1p_2...p_k)-1}-1)\Big]$$

is divisible by m. This is the generalization of Fermat's theorem to composite numbers. The "proof" is again as before. The first term comes from the totality of configurations less the translationally invariant ones. The second term subtracts the cyclic configurations with period (m/p_i), but in so doing, we have subtracted the configurations with period (m/p_ip_j) twice, because this is a subperiod in both the configurations with period (m/p_i) and (m/p_j). Therefore, to correct the sum we must add them once again. Now, however, we have added too much, because the configuration with period $(m/p_ip_jp_k)$ was first subtracted three times [in configurations (m/p_i), (m/p_j) and (m/p_k)] and then added three times [in configurations $(m/p_i\,p_j)$, $(m/p_i\,p_k)$, and $(m/p_j\,p_k)$] therefore we have to subtract it once more. And so on.... Finally, we obtain the expression shown, which represents the number of configurations free of any of the above subperiods. These remaining configurations may be classified into classes of m members each, and hence, this number must be divisible by m.

We are indebted to Persi Diaconis for bringing to our attention a somewhat similar proof of Fermat's theorem by Golomb.[2] However, the physical significance of the primes or possible extensions of the theorem to composite numbers are not discussed in this note.

In conclusion we have found an interesting analogy to the primes that is related to a lack of symmetry of certain physical systems.

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[1]C. S. Ogilvy and J. T. Anderson, *Excursions in Number Theory* (Oxford University, New York, 1966).
[2]S. W. Golomb, Am. Math. Mon. 63, 718 (1956).

Nonlinear effects in a simple mechanical system

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We describe a nonlinear mechanical system that is easy to construct and demonstrates most of the nonlinear effects associated with mechanical systems. The equation of motion for the system is easily derived through a geometrical argument and is found to be Duffing's equation. The relative strengths of the linear and nonlinear terms can be easily varied and it is possible, in principle, to make the linear term vanish completely. The system is also considered in a driven form. Periodic motions of the system are analyzed theoretically and the results are compared with experiment. Nonperiodic motions are also considered.

I. INTRODUCTION

Nonlinear effects are important in many areas of physics[1] and it is fairly safe to say that all physical phenomena become nonlinear when the relevant parameters are made sufficiently large. This paper presents a nonlinear mechanical system that is easy to construct and demonstrates most of the nonlinear effects associated with mechanical systems. The equation of motion for the system is easily derived through a geometrical argument and is found to be Duffing's equation. The relative strengths of the linear and nonlinear terms can easily be varied and it is possible, in principle, to make the linear term vanish completely. The system is also considered in a driven form. In Sec. II the system is described and its equation of motion derived. In Sec. III the periodic solutions of the equation of motion are obtained. The results of the experiment are compared with the theory in Sec. IV. Nonperiodic solutions are discussed in Sec. V.

II. APPARATUS

The system consists of a glider on an air track as shown in Fig. 1. A restoring force is supplied by a length of rubber (obtained by disecting a golf ball) that runs perpendicular to the axis of the air track in the horizontal plane. The ends of the rubber band are fixed at points symmetrically placed on opposite sides of the track while the midpoint passes through a vertical slot in the glider. Using the quantities labeled in Fig. 1, the equation of motion of the glider is[2]

$$m\ddot{x} = -2k\,[1-(l_0/d)(1+x^2/d^2)^{-1/2}]x,$$

where the relaxed length of the rubber band is $2l_0$ and its spring constant is $k/2$. (If one thinks of the system as consisting of two identical rubber bands they would have relaxed lengths l_0 and spring constants k.) If x^2/d^2 is much less than 1 this becomes

$$m\ddot{x} = -(2k\delta/d)x-(kl_0/d^3)x^3, \quad (1)$$

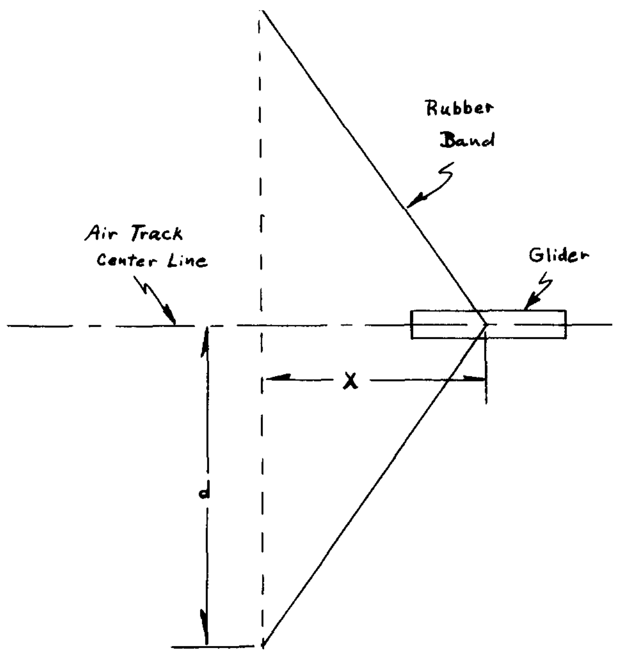

Fig. 1. Apparatus as seen from above.

where $\delta = d - l_0$. We now see how the relative strengths of the nonlinear and linear terms can be controlled by simply varying how lax the rubber band is in the equilibrium position.

A driving force is applied to the system by the interaction of a solenoid and a bar magnet. The magnet is held above the glider by a clamp that fixes one end of the magnet to the corresponding end of the glider. The solenoid is fixed with respect to the track so that the free end of the magnet is always within the solenoid. When a constant current passes through the solenoid, the force on the magnet is independent of position within a fairly broad range.[3] When a sinusoidally varying current with angular frequency ω passes through the solenoid we have a nonlinear driven oscillator and the equation of motion becomes

$$m\ddot{x} = -(2k\delta/d)x - (kl_0/d^3)x^3 + F\cos\omega t, \quad (2)$$

where F is proportional to the amplitude of the current and will be found experimentally.

III. MATHEMATICAL RESULTS

Equation (2) is of the form[4]

$$\ddot{x} = -\alpha x - \beta x^3 + G\cos\omega t. \quad (3)$$

We assume that β and G are sufficiently small so that the last two terms are of a higher order than the other terms in the equation. The apparatus can easily be adjusted so that this holds. When Eq. (3) is expressed as

$$\ddot{x} + \alpha x = -\beta x^3 + G\cos\omega t, \quad (4)$$

we see that either x must be small or the $\ddot{x}$ and αx terms must cancel to lowest order.

Since we seek a periodic solution we are free to expand x in a Fourier series. Without the nonlinear term this would be a driven harmonic oscillator and the solution would be of the form $A\cos\omega t$. Since β is small it seems reasonable to assume that the $A\cos\omega t$ term dominates the series. When the series is substituted into Eq. (4) and we require the coefficient of $\cos\omega t$ to satisfy the resulting equation we have

$$\omega^2 - \alpha = 3\beta A^2/4 - G/A. \quad (5)$$

This relates the frequency and the amplitude of the dominant term in the Fourier series expansion of x. Equating the corresponding coefficients of other terms of the series is expected to add higher-order corrections to x. The general features of Eq. (5) are represented in Fig. 2 as well as the free oscillations of the same system that is obtained by setting $G = 0$.

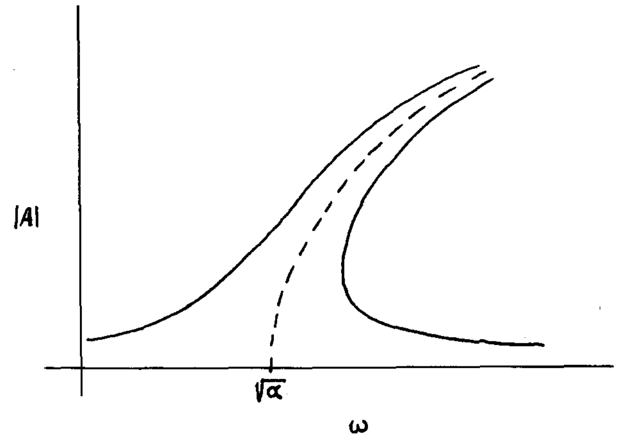

Fig. 2. Amplitude versus frequency for nonlinear oscillator with no damping. Dashed curve represents free oscillation. Solid curve is response for oscillator driven with fixed amplitude.

If a small amount of friction is included in the form of a $-c\dot{x}$ term in Eq. (2), the curve is approximately the same as the frictionless case except in the vicinity of the free oscillation solution. In that region the two branches are brought together in much the same fashion as in the damped linear case. This is sketched in Fig. 3.

As a function of frequency the curve shown in Fig. 3 is seen to be multivalued. This leads to a behavior that is fairly common in nonlinear mechanical systems called the "jump phenomena" and is easily seen in our system. If one, keeping the amplitude of the driver fixed, starts at a low frequency [(a) in Fig. 3] and slowly increases the frequency the amplitude will rise following the curve up to a maxi-

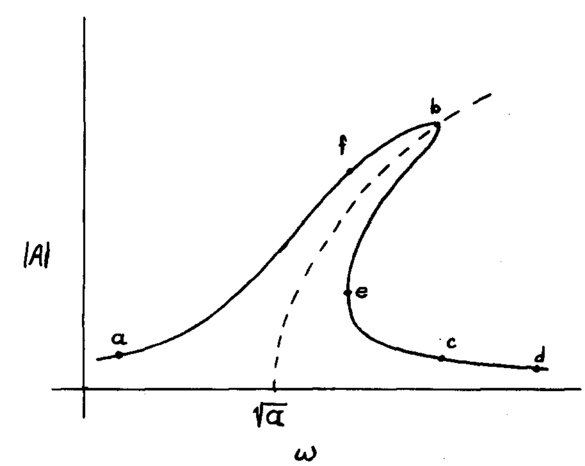

Fig. 3. Amplitude versus frequency for nonlinear oscillator with damping. Dashed curve represents free oscillation. Solid curve is response for oscillator driven with fixed amplitude.

mum [(b) in Fig. 3]. A slight increase will cause the amplitude for steady periodic motion to "jump down" to a smaller value [(c) in Fig. 3]. Further increase in the frequency will lead to a gradual decrease in the amplitudes [(d) in Fig. 3]. If the frequency is decreased from point c a gradual increase in the amplitude will be observed until we reach point e. If the frequency is decreased further, the amplitude for periodic oscillation will "jump up" to point f in Fig. 3. The underside of the curve between points b and e represent unstable periodic motions. These are shown to be parametrically unstable for the undamped oscillator in the Appendix.

Free oscillation of the extreme nonlinear case where $\alpha = 0$ can also be analyzed. Equation (3) then becomes

$$\ddot{x} = -\beta x^3, \tag{6}$$

which is a special case of

$$\ddot{x} = -\beta x^n \quad (n \text{ odd}). \tag{7}$$

This has an energy integral given by

$$\dot{x}^2/2 + \beta x^{n+1}/(n+1) = \beta A^{n+1}/(n+1),$$

where A is the amplitude of the oscillation. This can be written as

$$\frac{T}{2} = [(n+1)/2\beta A^{n+1}]^{1/2} \int_{-A}^{A} dx[1-(x/A)^{n+1}]^{-1/2}$$
$$= [(n+1)/2\beta]^{1/2} A^{(1-n)/2} \int_{-1}^{+1} du(1-u^{n+1})^{-1/2},$$

where T is the period. Since the integral is independent of amplitude we see

$$T \propto A^{(1-n)/2}$$

or

$$\omega \propto A^{(n-1)/2}.$$

For our system in the extreme nonlinear case we have

$$\omega \propto A. \tag{8}$$

IV. EXPERIMENTAL RESULTS

For our apparatus[5] $k = 3800 \pm 100$ dyn/cm, $l_0 = 14.09$ cm, $m = 263.6$ g, $d = 18.0$ cm, and $\delta = 3.9 \pm 0.1$ cm where only the significant uncertainties are given. With these values, α and β of Eq. (3) become 4.17 1/sec^2 and 2.62×10^{-2} 1/sec^2 cm^2, respectively. When x is a few centimeters we see that the third-order term is approximately 20% of the linear term. The next term (fifth order) can be shown to be approximately 10% of the third-order term. The damping term is found experimentally to be approximately $-6 \times 10^{-3}\dot{x}$. Hence our calculations for the driven, nonlinear oscillator where damping is neglected are applicable. For the driven case the amplitude of the driver divided by m [G in Eq. (3)] is 0.487 d/g. This is achieved with an alternating current of constant amplitude near 1 A. The theoretical curves and experimental points are shown in Fig. 4 and are in good agreement. The uncertainty in A is about 0.2 cm.

By scanning through the frequency range the "jump phenomena" is observed. The frequency at which the "jump up" occurs is in good agreement with theory as shown in Fig. 4. The frequency for the "jump down" is determined by the damping if the magnitude of the driving force is kept constant as discussed in Sec. III. With our apparatus, the magnitude of the driver falls off at large amplitudes and it is this effect that determines the exact frequency of the "jump down."

During the "jump" the motion is not periodic and appears to consist of a combination of periodic motions with a "beatlike" behavior. This eventually dies down to a purely periodic motion. The values given in Fig. 4 represent this

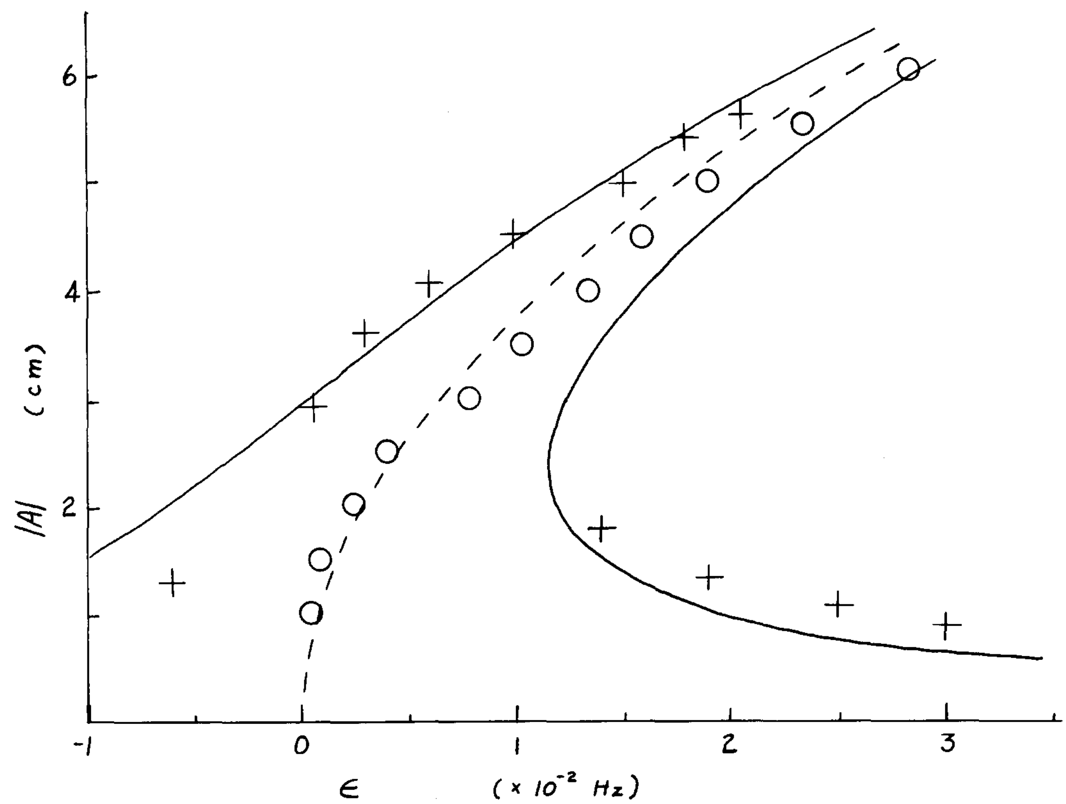

Fig. 4. Comparison of theory with experiment. Dashed curve is the theoretical prediction for free oscillation and the circles are the corresponding experimental results. The solid curve gives the theoretical results for the driven oscillator and the crosses are the corresponding experimental results.

periodic motion. This "beatlike" behavior also occurs when the driver is abruptly turned on with initial conditions $x = 0$ and $\dot{x} = 0$.

This would be the expected behavior for a linear system where, initally, natural and driving frequency components are present. In that case the natural frequency component dies away with time. The same argument cannot be applied to the nonlinear case due to the coupling of the periodic components via the nonlinear term. Nonperiodic solutions of Eq. (4) consisting of two frequency components will be found in Sec. V. The presence of these solutions seems to explain the "beatlike" behavior observed.

When the sinusoidal driver is replaced by a square wave or saw tooth driver of the same rms amplitude the results are remarkably similar to the earlier results showing an insensitivity to the additional harmonics in the driver.

The system was also set up in the extreme nonlinear case [$\alpha = 0$ in Eq. (3)] and free periodic oscillations examined. We saw general agreement with the $\omega \propto A$ law, but the results were unclear at low amplitude where the energy content of the system is very small and minor external disturbances become important. When driven, the system still exhibited the "jump phenomena," but no comparison with the above theory can be made since it is not applicable in this regime.

V. NONPERIODIC SOLUTIONS AND INITIAL CONDITIONS

The periodic solutions found in Sec. III contain no arbitrary constants that might be used to satisfy initial conditions. In the linear driven harmonic oscillator problem this is done by adding the general solution of the homogenous equation (free harmonic oscillator) to the solution of the driven oscillator. This approach is not applicable here since the sum will not be a solution.

Our approach is to start with an expression that satisfies the initial conditions and use the equation of motion to determine the newly introduced frequency. A particular example is presented in detail followed by a statement of the general result.

We seek a solution of Eq. (4) with initial conditions

$$x(t=0) = 0,$$

$$\dot{x}(t=0) = 0.$$

The expression

$$x = A\cos\omega t - A\cos\omega_0' t \tag{9}$$

satisfies the intial conditions and is taken as our approximate solution. A and ω_0' are to be determined by substitution into Eq. (4) and equating coefficients of $\cos\omega t$ and $\cos\omega_0' t$. It is assumed that the other frequency terms are of higher order. When this is done we obtain

$$A(\alpha - \omega^2) = -9\beta A^3/4 + G, \tag{10a}$$

$$\alpha - \omega_0'^2 = -9\beta A^2/4. \tag{10b}$$

These may now be solved for A and ω_0'. Rather than carry out the algebra we prefer to point out some key features. Equation (10b) is similar to Eq. (5) without a driver ($G = 0$), but ω_0' is shifted farther from $\sqrt{\alpha}$ than in the earlier case. This may be understood to some extent by noting that Eq. (9) does not have a well-defined amplitude and ranges from 0 to $2A$. Equations (10a) and (10b) may be combined to give

$$A = G/(\omega_0'^2 - \omega^2), \tag{11}$$

which is the usual resonance behavior where the natural frequency is ω_0'.

Since (10a) is cubic in A, the solution is, in general, not unique. In order to choose among the possible candidates we examine the higher-order terms to see if they are small in accordance with our assumptions (in particular, we must approximately maintain the initial conditions when the higher-order terms are added). For this case, we find we must take the solution A, of Eq. (10a), with the smallest absolute magnitude.[6]

More general initial conditions may be satisfied by starting with

$$x = A\cos\omega t + B\cos\omega_0' t + C\sin\omega_0' t.$$

Here B and C are to be adjusted in accordance with the initial conditions and may be expressed in terms of them. The resulting expression is substituted in Eq. (4). When coefficients of $\cos\omega t$, $\cos\omega_0' t$, and $\sin\omega_0' t$ are equated we have

$$A(\alpha - \omega^2) = -3\beta A\,[A^2 + 2(B^2 + C^2)]/4 + G,$$

$$(\alpha - \omega_0'^2) = -3\beta(2A^2 + B^2 + C^2)/4,$$

where the second equation is given twice. As in the first case, the first equation will give multiple solutions for A. Selection of a solution will depend on higher-order terms and will depend on the initial conditions.

The presence of the two frequency components obtained above seems to explain the "beatlike" behavior seen in the experiment. From our experimental observations we also know that ω_0' frequency terms are eventually dampened way leaving only the periodic behavior.

ACKNOWLEDGMENTS

The authors wish to thank George Moore and Brenton Stearns for their encouragement and helpful comments concerning this work.

APPENDIX

We examine the stability of the periodic solutions by substituting $x = x_0 + \epsilon$ in Eq. (3), where x_0 is a periodic solution and ϵ is small. This leads to

$$\ddot{\epsilon} + \alpha\epsilon + 3\beta x^2\epsilon = 0.$$

When x is approximated by $A\cos\omega t$, this becomes

$$\ddot{\epsilon} + (\alpha + \tfrac{3}{2}\beta A^2)\epsilon + \tfrac{3}{2}\beta A^2(\cos 2\omega t)\epsilon = 0.$$

This is the equation of a parametric oscillator with natural frequency $\omega_0^2 = \alpha + (3/2)\beta A^2$. Such systems are known to have unstable solutions $\epsilon(t)$ when[7]

$$\omega_0^2 - \tfrac{3}{4}\beta A^2 < \omega^2 < \omega_0^2 + \tfrac{3}{4}\beta A^2$$

or

$$\alpha + \tfrac{3}{4}\beta A^2 < \omega^2 < \alpha + \tfrac{9}{4}\beta A^2.$$

But, α, β, and ω are related by Eq. (5) and we see that our solutions are unstable for

$$0 < -G/A < \tfrac{3}{2}\beta A^2.$$

If one is careful about keeping track of the sign in Eq. (5), it is easily seen that this corresponds to the under side of the response curve in Fig. 2 from the point where the slope is infinite out to $|A| = \infty$.

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[1]W. Heisenberg, Phys. Today **20**, (5), 27 (May 1967).
[2]Systems similar to the one present here have been discussed to some extent by Stockard, Johnson, and Sears, Am. J. Phys. **35**, 961 (1967); and Thomchick and McKelvey, Am. J. Phys. **46**, 40 (1978).
[3]Experimentally the force is found to vary by less than 10% for amplitudes of 3 cm and less than 20% for amplitudes of 6 cm.
[4]This is known as Duffing's equation. Periodic solutions of Duffing's equation are also treated in J. J. Stoker, *Nonlinear Vibrations* (Interscience, New York, 1950), Chap. IV.
[5]Of course, our rubber band is not an ideal spring. This data was obtained by considering the characteristics of the rubber band when stretched over the range corresponding to its enlongation when x varies from 0 to 6 cm.
[6]If $x = A(\cos \omega t - \cos \omega_0' t) + B_1 \cos(2\omega - \omega_0')t + B_2 \cos(2\omega_0' - \omega)t$ is used in place of Eq. (9) one finds

$$B_1/A = (\omega_0'^2 - \alpha)/\{3[\alpha - (2\omega - \omega_0')^2]\}$$

and

$$B_2/A = (\omega_0'^2 - \alpha)/\{3[\alpha - (2\omega_0' - \omega)^2]\}.$$

From the relations given in Eq. (10), one can see that both of these will be small provided we choose the solution A of Eq. (10a) with the least absolute magnitude when there is more than one real solution. The algebra involved in this derivation is fairly lengthy.
[7]W. Case, Am. J. Phys. **48**, 218 (1980); or Landau and Lifshitz, *Mechanics*, 3rd ed. (Pergamon, Oxford, 1976).

Relating mystical concepts to those of physics: Some concerns

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Some of the present attempts to emphasize the parallels between mystical concepts and those of physics are listed with a discussion of the reasons for a possible interaction between physics and mysticism. Difficulties with these approaches are considered; some of the related educational or learning aspects for physics are also discussed. It is concluded that efforts to use mysticism to help understand physics are not justified and alternative pedagogical techniques are considered.

I. INTRODUCTION

In recent years there has been a marked increase in the interest in and discussion of the similarities between physics and mysticism, particularly Eastern mysticism.[1–6] Although scientists have intermittently observed parallels between the two disciplines, the marked increase in the discussion of such parallels is an occurrence of the last half-century, and more particularly of the last decade.[7]

This paper asserts that the reasons advanced for fostering an interaction between physics and mysticism and for discussing their respective concepts have not been adequate nor always appropriate to justify such an interaction. After a brief overview in Sec. II of the most popular correlation theories, this discussion will have two broad concerns. Section III will examine the difficulties inherent in and overlooked by the dialog of mysticism and physics as it has so far proceeded. In particular, it will examine the idea of a "common mysticism" and an apparent argument for including mystical concepts in discussions of concepts in physics. Sections IV and V will examine the learning and teaching of physics as it possibly relates to mystical concepts. The sections will address some of the questions relative to teaching the nature and content of physics and will suggest alternatives to the inclusion of mystical concepts in physics-teaching or -learning.

II. REASONS FOR THE DIALOG

The reasons for mixing physics and mysticism in books and courses have been given elsewhere by Harrison,[8] and certainly they have a strong appeal. Anything that can help to make the principles of physics more readily available to a larger public is a potentially valuable contribution. Yet, the specific reasons for the dialog between mysticism and physics have not been adequately developed. The reasons seem largely to fall under the need for understanding physical concepts. That is, to say, attempting to understand the new concepts in physics or at least learning to become comfortable with them appears to be the guiding force in the appeal of mysticism for both layman and physicist. In fewer than a hundred years, physics has changed from a world view with concepts that were mechanistic, deterministic, and largely absolute, to a world view employing concepts that are relative, often nondeterministic, and stochastic in nature. Furthermore, and as a consequence of this change, it is now recognized that the experimental result can never be totally isolated from the actions of the experimenter. The former, older world view, the Newtonian world view with its Greek bases and with its unique intellectual bias, has been usurped in fewer than 75 years. Although our Western culture had similar beginnings to those of physics, it is possible that physics has in those 75 years felt isolated or