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The shape of "the Spandex" and orbits upon its surface

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What is the shape that results when a flat rubber sheet is warped by placing a heavy ball upon it? We show that, at distance R far from the center of a ball of mass M, the height h of the surface above the ball's center is given by $h(R) = AM^{1/3}R^{2/3}$, where A is a constant determined by the stretchiness of the rubber and the earth's gravitational constant. This happy result allows one to analyze the orbits of marbles and coins as they roll across the surface in some detail, providing very nice analogues for a wealth of topics in celestial mechanics, from Kepler's laws to tides and the Roche limit. © *2002 American Association of Physics Teachers*.

Contents

I. INTRODUCTION

This adventure began in the grocery store (or perhaps a hands-on science museum), where one of the authors (GDW) remembers being fascinated by the peculiarly shaped plastic surface on which donated coins would roll and eventual spiral down into the collection bin. He wondered about the mathematical form of the shape of this surface, and how accurately these rolling coins modeled planetary orbits. Later, in his first attempt at teaching astronomy, his inability to fully understand the textbook discussions of the tides[1] led him to pursue the development of a device to demonstrate tidal effects. By stretching some elastic material across a square wooden frame (later, a circular frame was fashioned) and suspending a heavy mass from the middle of the stretchy cloth, a surface similar to that in the grocery check-out aisle was produced. (This material, which goes by the name spandex, is often used to make biking shorts, and is readily found in fabric shops.) This construction, while allowing marbles and coins to roll reasonably smoothly across its surface, had an advantage over its science museum and grocery store counterparts—its surface was flexible, becoming steeper with heavier masses, thus nicely mimicking gravitational potential surfaces. Promptly christened "the Spandex," this device has since been used in various educational settings to demonstrate a variety of astronomical effects, especially tidal and orbital phenomena.[2] The Spandex has also been a great research instrument for beginning students, especially helpful in demonstrating features from topics such as the formation of the solar system, orbits around binary systems, circular versus elliptical orbits for planets, and the origins of Cassini's Division and the Kirkwood Gaps.[3]

One moral of this story is that a little knowledge is a dangerous thing. One of us (GDW) had read a little bit about minimal surfaces and clung to the notion (in spite of mounting contradictory evidence provided by his students) that the Spandex was like a soap bubble film. Thus, he kept insisting that the shape of the Spandex should be the same as the shape of a soap bubble film suspended between two horizontal rings sharing the same vertical symmetry axis, the upper ring large (about a meter across) and the lower ring very small. The soap bubble shape that minimizes energy between two hoops so arranged has been long established to have a profile described by the catenary curve [$\sim A \cosh(x/A)$, but rotated a quarter turn from its usual orientation in which it describes a string suspended between two points]. Only in the face of a mound of data showing that the height of the cloth above the hanging mass was in fact a power law, $h(R) = AM^{1/3}R^{2/3}$, did we finally seek a theoretical description of the surface not based on soap bubbles' mathematical description.[4]

One might alternatively suppose that the Spandex surface satisfies LaPlace's equation, like a static, weighted drumhead. Then one would expect the surface height to be given by the form $k \ln(r) + C$, and the orbits would be the gravitational orbits about a very long stick of mass. We show below though, that the Spandex only satisfies LaPlace's equation when the tangent to the surface is nearly vertical, a region, incidentally, that we did not pursue experimentally. One might next turn to the bulk of the literature on elasticity of surfaces, however, a perusal of a few of these textbooks turned up no direct treatment of this topic. For example, Love's text[5] gives an expression for the vertical displacement of an initially horizontal circular plate, supported on its boundary and then loaded at its center. We were not able to extract our 2/3 power from a limiting case of this expression (which involved quadratic and logarithmic terms); apparently a flexible surface like the Spandex is not a special case of the more rigid plate problem.

Finally, to close this section on related literature, we mention some interesting articles from this journal. The most recent concerns objects rolling on cylindrically symmetric surfaces, discussing stability of orbits and concavity.[6] Others include interesting discussions of the tides and/or of orbital problems and references.[7,8,9]

In this paper, in addition to reporting on the shape of the Spandex in Sec. II, we also present some experiments concerning orbits on the Spandex in Sec. III, primarily the analogue to Kepler's third law (KIII) for planetary orbits. One can, with some care, determine how the period depends upon the radius for near circular orbits to compare the results of Spandex orbits to those of an inverse square force law. We have performed this experiment about a half-dozen times with groups ranging from junior high school students to senior-level physics majors, and have observed directly the benefits of having a contrast to KIII on the actual understanding of KIII. Since the data from the Spandex do not conform to the usual statement of KIII, T^2/R^3 = constant (the Spandex more nearly follows T^3/R^2 = constant, instead), the students are better able to internalize the meaning of KIII, in our opinion, by having an example of when it is not true.

II. THE SHAPE OF THE SPANDEX

In Fig. 1 we show how the profile of the Spandex changes as the hanging mass is increased. This version of the Spandex was made from a standard 4 ft×8 ft piece of $\frac{3}{4}$-in-thick plyboard. We cut a large circular hole, with radius a little more than a half meter, and cut and hinged the plyboard lengthwise, so that it would be easier to transport. Then we attached the spandex material, using dozens of tacks around the circular border. In Fig. 2 we consolidate the data of Fig. 1, plotting $\ln(h)$ vs $\ln(MR^2)$. The data fit a line with slope 1/3 very nicely, thus exhibiting the relationship between the height of the surface, the hanging mass, and the distance from the center axis,

$$h(R) = AM^{1/3}R^{2/3}. \quad (1)$$

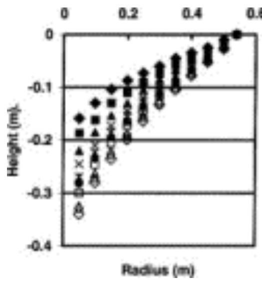

Figure 1.

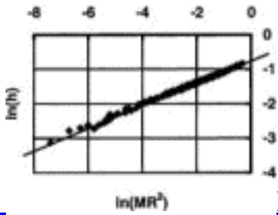

Figure 2.

One can also arrive at this result theoretically under the assumption that the Spandex be modeled as a collection of concentric, massless rigid rings connected to each other by means of massless springs with spring constant k. We also assume that the number of springs attached to the inside of each ring is proportional to the circumference of the ring. Newton's second law (see Fig. 3) gives

$$n_0(F_{\text{spring}})_0 \sin(\theta_0) - Mg = 0, \qquad (2)$$

for the lowest ring (where n_0 is the number of springs attached to the hanging mass M) and

$$-n_i(F_{\text{spring}})_i \sin(\theta_i) + n_{i+1}(F_{\text{spring}})_{i+1} \sin(\theta_{i+1}) = 0 \qquad (3)$$

for the ith ring. Thus, for any ring, not just the first one, the equilibrium condition is

$$nF_{\text{spring}} \sin(\theta) = Mg, \qquad (4)$$

where the subscripts have been suppressed.

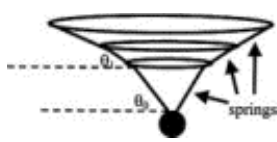

Figure 3.

Suppose that each spring has equilibrium length equal to the spacing of the rigid rings, dR, and that δs is the amount each ideal spring gets stretched after the mass is suspended. Then from the triangle (see Fig. 4) δs must satisfy $\delta s = dR(1/\cos(\theta)-1)$. Furthermore, the number of springs holding up each ring is proportional to the circumference, so $n = \lambda 2\pi R$ where λ is the same constant for every ring. The equilibrium condition becomes

$$(\lambda 2\pi R)[k\,dR(1/\cos(\theta) - 1)]\sin(\theta) = Mg \qquad (5)$$

or

$$R = BM/(\tan(\theta) - \sin(\theta)), \qquad (6)$$

where $B = g/(2\lambda\pi k\, dR)$ is small if the Spandex does not stretch easily and large if it is easy to stretch. We take the limit as dR goes to zero in such a way as to leave B constant. This is natural since halving the length of a spring doubles its spring constant.

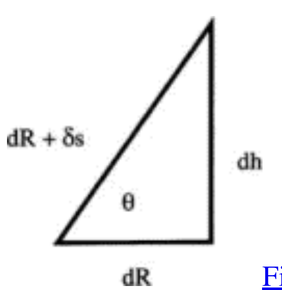

Figure 4.

In our experiments most of the action on the Spandex occurred in the regions relatively far from the center, where the angle θ is small. Since $\tan(\theta)$ is the rate of change of h with respect to R, then Eq. (6) is the differential equation defining the shape of the surface of the Spandex. To avoid attempting to solve this rather nasty nonlinear differential equation exactly, we take the small angle limit. In the limit of small θ the denominator simplifies to

$$\tan(\theta) - \sin(\theta) \rightarrow (\theta + \theta^3/3 + \cdots) - (\theta - \theta^3/6 + \cdots) \rightarrow (\theta^3/2 + \cdots) \quad (7)$$

or since

$$dh/dR = \tan(\theta) \rightarrow \theta \quad (8)$$

we can write

$$R = 2BM/(dh/dR)^3. \quad (9)$$

Integration completes this theoretical derivation of the claimed result for small θ (which is equivalent to large distances), $h(R) = AM^{1/3}R^{2/3}$, where $A = 1.5(2B)^{1/3}$. Qualitatively, this form has features that one would expect, such as stretchier cloth (meaning larger A) implies more vertical displacement (larger h); also heavier hanging mass implies more vertical displacement.

We also note that Eq. (6), in the limit as θ approaches $\pi/2$, yields $dh/dR = c/R$, which is precisely the form of the solution to LaPlace's equation with axial symmetry. Thus deep inside of the well, one might expect to see behavior similar to that of planets or stars orbiting a very long stick shaped galaxy, or electrons orbiting a long straight positively charged wire. We did not investigate marble orbits very near the center experimentally, however.

III. CIRCULAR ORBITS ON THE SPANDEX

We now turn to the experimental determination of the behavior of circular orbits on the Spandex. While the variance of the data is rather large since the orbital radius decreases significantly during one orbit, repeatable results are obtainable, even with inexperienced experimenters. Our simple theoretical model, which will utilize only point masses sliding without friction, suffices to describe these orbits despite the fact that, experimentally, we always used rolling objects. Curiously, we have never noticed significant differences in the period of the orbit for a marble versus a coin, or for large versus small marbles, despite our varied prejudices. Even with rather large uncertainties in the data, one might be tempted to characterize the data shown in Fig. 5 by saying that T^3/R^2~constant (instead of T^2/R^3 = constant, as in Kepler's third law), especially after the analysis that follows. We use the shape of the Spandex obtained above and apply Newton's second law.

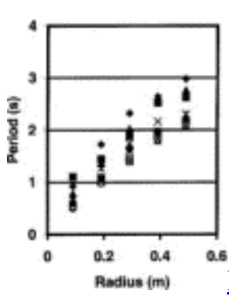

Figure 5.

Figure 6 shows a side view of the Spandex, with forces as shown. Applying Newton's second law to an object sliding on the Spandex without friction in a horizontal circular orbit of radius R, one gets $N\cos(\theta) - mg = 0$ for the vertical component, where N is the normal force that the Spandex exerts on the object. The horizontal component of the second law then becomes

$$mg\tan(\theta) = mV^2/R. \quad (10)$$

Using $\tan(\theta) = dh/dR = (2/3)A(M/R)^{1/3}$ and $V = 2\pi R/T$, we get a prediction for the analogue to KIII on the Spandex, i.e.,

$$g(2/3)A(M/R)^{1/3} = (2\pi)^2 R/T^2 \qquad (11)$$

or

$$T^3/R^2 = (2\pi)^3(2gA/3)^{-3/2}/M^{1/2}. \qquad (12)$$

Thus, the Spandex is seen to reverse the exponents on R and T (for fixed M) when compared to Kepler's third law.

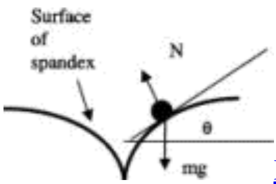

Figure 6.

We close this section with Fig. 7, which rearranges the data in Fig. 5 to show $\ln(T)$ plotted versus $\ln(R^2/M^{1/2})$. The data match a line with slope 1/3 as expected, at least in the upper right region where the small angle condition is satisfied. The y intercepts of the lines in Figs. 7 and 2 both yield consistent values for the constant A~0.5 $(\mathrm{m/kg})^{1/3}$.

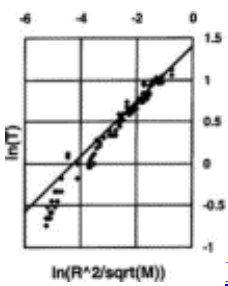

Figure 7.

IV. CLOSING

In conclusion, we assert that the Spandex is worth the time to build and use to demonstrate orbital phenomena at any level, even without the satisfying agreement achieved above between the theoretical models and the experimental results. Pedagogically, seeing precession, measuring periods versus radii, and investigating tides and orbits in potential well models is valuable whatever the mathematical sophistication of the student. A further benefit that should not go unmentioned is that it is difficult to keep even the most science-averse students from taking a chance and rolling a marble or two. [See Fig. 8, which shows a small flock poring over an early version of the Spandex with two hanging masses attached, allowing study of orbits about a (nonrotating) binary system.] The fact that the surface does not exactly match reality is not necessarily a hindrance; rather, we would argue that student (and teacher) understanding is enhanced (beyond that which would occur if one had a perfect small scale model of a rotating binary gravitational system) by a dialogue like the following:


Figure 8.

Teacher: Now, what are the contrasts between orbits on this crude binary model and orbits about an actual binary star system?

Student 1: The Spandex model is too small! Student 2: Duuuhhhh! . . . Hey, and planets don't roll.

Student 3: . . . and there's way too much friction.

Teacher: Good, anything else?

Student 2: The Spandex bounces when I pull it up and let it go, like this.

Teacher: Um, right . . . anything else?

Student 3: The shape of the Spandex is not quite right, right?

Imaginary ideal student: Gravity has an inverse square force law, implying that the surface height should go as $1/R$, while the Spandex has an inverse cube root force law at large distances, corresponding to a surface behavior of $R^{2/3}$, right?

Teacher: Very good, and what about the dependence of the orbits on the center mass?

Student 3: . . . bigger masses at the center mean faster orbits and more tidal effects.

Imaginary ideal student: Right, in fact if you want to half the orbital time at fixed distance in real gravity, you increase the central mass by a factor of 4, but with the Spandex you have to increase the central mass by a factor of 64!

Teacher: Uh . . . , why don't we all get out our pencils and check that? . . . By the way, can you think of another contrast involving the motion of these two binary systems?

Imaginary ideal student: Yes, we would have to center the Spandex on a big turntable, rotating at just the right speed to more accurately represent an equal mass gravitational binary system.

Teacher: . . . and what about an unequal mass system?

Student 2: Let's try it! I'll pick up this end over my head and you . . .

A discussion like this, idealized or not, provides numerous opportunities for the teacher to "spiral back," and better helps students to understand qualitatively how the universe behaves and how it does not. Quantitatively, besides giving opportunities for novice students to do science, graphs, data analysis, etc., it shows what the universe is not like, giving students a chance to reflect on the differences and thereby better enhancing their chances of retaining the material.[10] The Spandex is not unique in this respect; other astronomical models afford similar opportunities. For example, students attempting to build a scale model of the solar system eventually come to a better appreciation of the vast emptiness of space than they would by less interactive methods of discovery, and a commercial orrery generally exhibits some deficiency in its representation of the coincident motion of the planets and moons that can occasion a pedagogically valuable experience. Incidentally, the notion of an "ideal imaginary student" while perhaps amusing to envision, and appealing at first glance, can actually be subversive in a subtle way. To illustrate, we note that we scarcely can resist launching into an elevated discussion with an exceptional student who vocalizes deep insights in the classroom. But it is our considered opinion that, as gratifying as this kind of discussion can be for the two participants, it rarely develops into effective pedagogy for the rest of the class.

ACKNOWLEDGMENTS

We would like to acknowledge the contributions of the students at Northwestern State University of Louisiana (NSU) who participated in the various outreach programs and measurements involving the Spandex, especially Chris Gresham, Randall Gauthier, Brian Dixon, Kristen Russell, Matt Creighton, and Michael Williams. In addition, we would also like to thank the administrators of the NASA/JOVE program, both at NSU and at NASA, whose initiatives allowed us to start this project. Finally we would like to thank the Society of Physics Students (SPS) and its various undergraduate programs that helped to sponsor

some of these efforts.

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3. For example, undergraduates R. Gauthier and C. Gresham presented their Spandex resonance demonstration exhibiting the cause of Cassini's Division at the AAPT 1995 summer meeting in College Park, winning the Society of Physics Students' Lecture Demonstration Award. Later, at the 1996 summer meeting in Denver, C. Gresham and students won the AAPT Video Competition with their presentation of the Kepler's third law analog using the Spandex. This paper, in fact, is a summary of the experimental results obtained by students (from junior high to college) over the last few years, with a little theory added. first citation in article
4. Frank Morgan, *The Math Chat Book* (The Mathematical Association of America (MAA), Washington, DC, 2000). Professor Morgan was kind enough to explain to GDW the difference between soap bubble films and the Spandex (soap bubble films do not obey Hooke's law, but rather the tensile force is constant with extension) after his entertaining presentation at the 2001 Louisiana Mississippi regional MAA meeting. In this book, and especially in his recent article, "Proof of the Double Bubble Conjecture," Am. Math. Monthly 8, 3 (March 2001), he describes his undergraduate students' remarkable contributions to the solution of this problem. first citation in article
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8. Aaron K. Grant and Jonathan L. Rosner, "Classical orbits in power-law potentials," Am. J. Phys. **62** (4), 310–315 (1994). first citation in article
9. Ralph Baierlein, "Rubber asteroids: Some orbits in introductory physics," Am. J. Phys. **62** (4), 378–379 (1994). first citation in article
10. These last paragraphs express strongly held opinions of the authors, but the origin and actual documentation of the value of "spiralling back" and of inaccurate physical models is provided in the text by Arnold Arons, *Teaching Introductory Physics* (Wiley, New York, 1996), along with a lot more good information for science teachers. first citation in article

CITING ARTICLES

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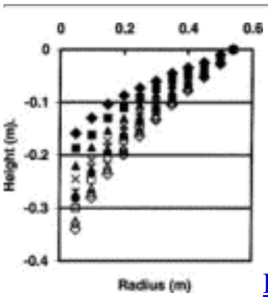
1. Trajectories of rolling marbles on various funnels
   L. Q. English *et al.*, Am. J. Phys. **80**, 996 (2012)

FIGURES



Full figure (7 kB)

Fig. 1. Measurements of the profile of the Spandex, a circular elastic sheet, supported by its boundary with a heavy mass hanging from the center of the sheet. The hanging mass is varied from 0.25 kg (closed diamonds) to 2.5 kg (open diamonds) with steps of 0.25 kg. First citation in article

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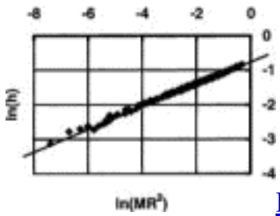

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Fig. 2. Data from Fig. 1 plotted as $\ln(h)$ vs $\ln(MR^2)$, where h is the height above the lowest point of the Spandex at a radial distance R when mass M is hung from its center. The line (not a fit) has slope 1/3 as predicted by the theoretical model. First citation in article

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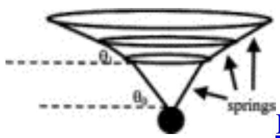

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Fig. 3. We model the Spandex as rigid, massless, concentric rings connected by massless springs; the number of springs attached to the inside of each ring is proportional to the circumference. First citation in article

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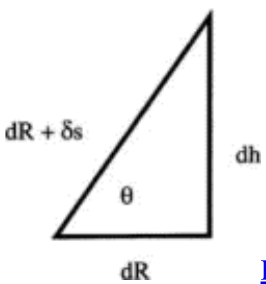

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Fig. 4. Each spring has an original length dR and stretches an amount δs. First citation in article

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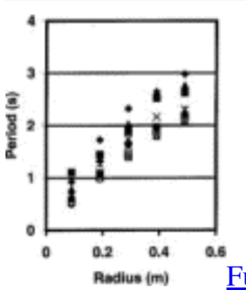

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Fig. 5. Experimental data for near circular orbits (analogous to Kepler's third law) on the Spandex for hanging masses varying from 0.25 kg (closed diamonds) to 2.25 kg (open triangles). Uncertainties in these measurements can be as much as 20%–30%, but notice that the concavity, though slight, is opposite that of Kepler's third law. First citation in article

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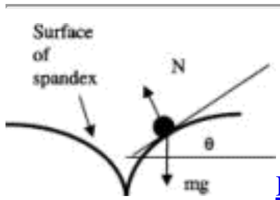

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Fig. 6. Diagram for applying Newton's second law to an object sliding without friction in a circular orbit on the Spandex. First citation in article

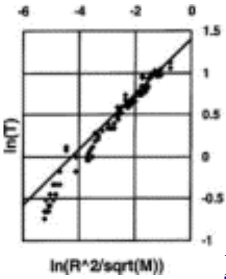

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Fig. 7. Data from Fig. 5. Plotted as $\ln(T)$ vs $\ln(R^2/M^{1/2})$, where T is the period for near circular orbits. The line (not a fit) has slope 1/3 and represents the prediction from Newton's second law in the limit of small tangent angles for the Spandex. The y intercept of this line corresponds to $A = 0.5\ (\mathrm{m/kg})^{1/3}$, as does the line in Fig. 2. First citation in article


Full figure (29 kB)

Fig. 8. Scientists intently preparing for orbital experiments about a binary system using an earlier rectangular Spandex. There are two masses hanging from the cloth, separated by about 1 m, and causing the indentations into which the balls eventually descend. One author (GDW) recently visited The Franklin Institute in Philadelphia, PA, and was surprised to see a similar surface (albeit rigid) that clearly had been in use for years to demonstrate orbits around binary systems. First citation in article

FOOTNOTES

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