The pendulum—Rich physics from a simple system

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We provide a comprehensive discussion of the corrections needed to accurately measure the acceleration of gravity using a plane pendulum. A simple laboratory experiment is described in which g was determined to four significant figures of accuracy.

I. INTRODUCTION

The simple pendulum is of historic and basic importance. Its approximate isochronism, discovered by Galileo, makes it an accurate and simple timekeeper and, in the hands of Newton, resulted in the first evidence that inertial and gravitational masses are proportional. Until relatively recently the plane pendulum provided the most accurate and convenient method for measuring the local gravitational acceleration. These acceleration measurements provided the earliest information on the shape and mass distribution of the earth.[1, 2, 3, 4, 5]

It is in the determination of the local gravity that the pendulum is usually employed in the student laboratory. One of the most interesting aspects of such a simple system is the rich variety of physics involved when the ideal pendulum is compared to a real experiment. It should be emphasized that as higher accuracy is demanded of an experiment, the theoretical description of the apparatus must become more realistic. In many cases the physics involved in this more sophisticated picture is at least as, if not more, interesting than the original idealized case. In this article we illustrate the analysis with the results of an actual laboratory experiment and describe the corrections necessary to attain an accuracy of one part in 10 000 for the acceleration of gravity.

II. EXPERIMENTAL PROCEDURE AND RESULTS

For a point pendulum supported by a massless, inextensible cord of length l the equation of motion for oscillations in a vacuum is

$$\ddot{\theta} + (g/l)\sin\theta = 0, \tag{1}$$

where $\dot{\theta} \equiv d\theta/dt$. For infinitesimal displacements we replace $\sin\theta$ by θ and the motion is simple harmonic with period

$$T_0 \equiv 2\pi(l/g)^{1/2} \equiv 2\pi/\omega_0. \tag{2}$$

If the cord length and the period are known, we can solve for the acceleration of gravity

$$g = 4\pi^2 l/T_0^2. \tag{3}$$

The fundamental experimental problem is then to measure l and T_0.

To attain a given precision in the acceleration of gravity, one must know the required precision for the measurements of length and time. The errors in g resulting from an error in l or T_0 may be estimated from the relations $\sigma_g = (4\pi^2/T_0^2)\sigma_l$ and $\sigma_g = (8\pi^2 l/T_0^3)\sigma_{T_0}$, respectively. The total error is the square root of the sum of the squares of the individual errors. The period measurement is relatively easy since the result is cumulative. For a precision of one part in 10^4, with a pendulum 3 m long, the time for 100 oscillations must be measured to within 0.02 s. The limiting measurement is the cord length. In order to achieve one part in 10^4 precision a cord length of 3 m must be measured to a precision of 0.3 mm, near the practical limit for the student laboratory.

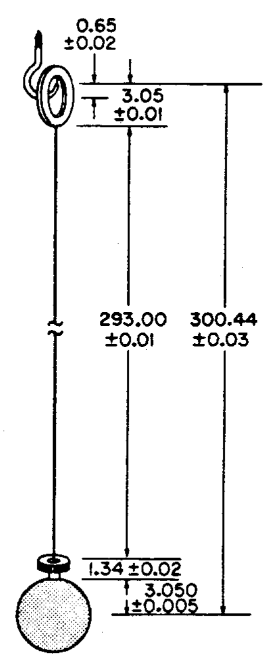

Fig. 1. Dimensions of the pendulum apparatus. Values are given in cm.

For our experiment we used the Leybold "ball with pendulum suspension"[6] shown in Fig. 1. The pendulum ball was 6.10 ± 0.01 cm in diameter, having mass 856.7 ± 0.1 g and suspended by a fine, stranded steel wire. The pendulum suspension consists of a small ring fit with a needle facing inward that rests in a socket at the end of a screw hook attached to the ceiling. The wire passes through a hole in a small screw cap on top of the ball. The length of the wire was measured in its static position with the aid of a second wire trimmed until its ends were exactly flush with the bottom of the ring and the top of the cap and then measured to the nearest tenth of a millimeter using a pair of 2-m sticks placed end to end. The dimensions of the ball, ring, and cap were measured with vernier calipers. The total distance between the point of support and the center of the ball was 3.0044 ± 0.0003 m The time for 100 oscillations was measured for ten trials using a handheld electronic stopwatch which provided digital readings to the nearest hundredth of a second. The principal sources of error were human reaction time and judgment of the instant when the pendulum reached the end of its swing. The measurements are recorded in Table I. The experimental period for the ten trials is

$$T = 3.478\,80 \;\pm 0.000\,17 \text{ s} \tag{4}$$

Table I. Measurements of pendulum period.

Trial	Time for 100 oscillations 100 T (s)	Period T (s)
1	347.84	3.4784
2	347.87	3.4787
3	347.86	3.4786
4	347.91	3.4791
5	347.91	3.4791
6	347.94	3.4794
7	347.84	3.4784
8	347.81	3.4781
9	347.84	3.4784
10	347.98	3.4798

If T were equal to T_0 the corresponding acceleration due to gravity according to Eq. (3) would be

$$g = 9.8007 \pm 0.0014 \text{ m/s}^2 . \tag{5}$$

However, once the length and period are measured, a large number of corrections must be applied since any actual pendulum deviates from the idealized assumptions underlying Eq. (3).

III. PENDULUM CORRECTIONS

If a given real pendulum results in a period of oscillation T, we may write

$$T = T_0 + \Delta T = T_0(1 + \Delta T / T_0), \tag{6}$$

where T_0 is the idealized period of Eq. (2). We may still use Eq. (3) to compute g if we use the equivalent ideal pendulum period from Eq. (6),

$$T_0 = T - \Delta T \simeq T(1 - \Delta T / T_0). \tag{7}$$

In applying Eq. (7), T is the measured period and ΔT is the theoretical correction relating the actual pendulum to the ideal pendulum. There are many such corrections depending on the desired level of accuracy. The corrections for finite amplitude and finite mass distribution are well known and we shall consider them only briefly; the effects of the air are not as well known and shall be discussed in greater detail. It is also necessary to consider elastic corrections due to wire stretching and motion of the support.

A. Finite amplitude correction

An exact analytic solution[7] to Eq. (1) involves the Jacobian elliptic sine function. The period may be expressed in terms of the complete elliptic integral of the first kind.[8] For small angular displacements Eq. (I) can also be solved by a perturbation expansion.[9] The correction to the period is

$$\begin{aligned}\frac{\Delta T}{T_0} &= \sum_{n=1}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 \sin^{2n}\left(\frac{\theta_0}{2}\right), \\ &= \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 \cdots\end{aligned} \tag{8}$$

where θ_0 is the maximum angular displacement in radians. For all of our measurements the amplitude was 3.0° ± 0.3°, which was measured with a large demonstration protractor. The effect of this correction is to make the ideal period longer by $\Delta T = 596\ \mu$s. In addition to lengthening the period, the finite pendulum displacement introduces an admixture of higher harmonics which can be observed by a Fourier analysis of the time-dependent displacement.[10]

B. Mass distribution corrections

A real pendulum bob has a finite size and the suspension wire has a mass. In addition, as in our case, the wire connections to the bob and the support may have some structure. All such effects are encompassed in the physical pendulum equation

$$T = 2\pi (I / Mgh)^{1/2} \tag{9}$$

where I is the total moment of inertia about the axis of rotation, M is the total mass, and h is the distance between the axis and the center of mass.

1. Uniform spherical bob of radius a

For the bob $I = ml^2[1 + (2/5)(a/l)^2], M = m$ and $h = l.$ Thus

$$\Delta T / T_0 = (1/5)(a/l)^2 \tag{10}$$

For a = 3.05 cm and l = 300.44 cm this correction leads to $\Delta T = +72\ \mu s$

2. Wire connections

In our case a small ring supported the pendulum at the upper end. For the bob and ring $I = ml^2 + I_r$ and $Mh = ml + m_r d_r$, where m_r is the mass of the ring, I_r is its moment of inertia, and d_r is the distance between the pivot at the end of the needle and the center of the ring. Thus the ring requires a correction

$$\frac{\Delta T}{T_0} = -\left(\frac{1}{2}\right)\left(\frac{m_r}{m}\right)\left(\frac{d_r}{l}\right) + \left(\frac{1}{2}\right)\frac{I_r}{ml^2}. \tag{11}$$

The values of m_r and d_r are 65.0 g and 0.65 cm, respectively. Also, the ring has an inner radius r_i = 1.75 cm and an outer radius r_o = 2.40 cm, and so $I_r = m_r[d_r^2 + \tfrac{1}{4}(r_i^2 + r_o^2)] = 171\ \mathrm{g \cdot cm^2}$. Therefore, Eq. (11) gives $\Delta T = -282 \mu s$.

A small cap attached the wire to the pendulum bob. The cap requires a correction

$$\Delta T / T_0 = -\left(\tfrac{1}{2}\right)(m_{c/m})(d_c / l), \tag{12}$$

where m_c *is* the mass of the plastic cap, not including its threaded screw, and d_c is the distance between its center of mass and the center of the bob. For m_c = 4.4 g and d_c = 3.9 g we obtain ΔT = - 116 μs. Similarly, the difference in densities between the aluminum screw and the steel bob produces a mass deficit m_s = - 6.1 g at a distance d_s = 2.3 cm and implies a correction ΔT = + 95 μs.

3. Mass and flexibility of wire

For the bob and wire of mass $m_w, I = (m + \tfrac{1}{3} m_w) l^2$ and $Mh = (m + \tfrac{1}{2} m_w) l$. The period correction to first order in the mass ratio m_w / m is

$$\Delta T / T_0 = -(1/12)(m_w / m) \tag{13}$$

Hence the finite mass of the wire reduces the period. To this level of precision it does not matter whether the wire acts as a rigid rod or as a flexible string.[11, 12, 13] For m_W = 1.367 g and m = 856.7 g we obtain ΔT = - 463 μs.

By an extension of the analysis of Ref. 11 we find that the second-order correction for a flexible wire is $\Delta T / T_0 = +(71/1440)(m_w / m)^2$. However, for a rigid rod it is $\Delta T / T_0 = (55/1440)(m_w / m)^2$ by Eq. (9). Thus if the cord is flexible the period is slightly greater than if it is rigid. In our case these terms give ΔT = + 0.44 and + 0.34 μs, respectively, and the difference is + 0.10 μs.

For completeness we should comment that when a flexible string connects the various parts of the pendulum it is possible that higher modes of motion of the system might be important. For the bob, its rotation about the point of contact with the wire will be most important.[14] The effect of this rotation correction is to multiply Eq. (10) by the factor $1 + (2/5)(a/l)$, a negligible correction in most cases since the correction of Eq. (10) itself is usually small. For our experiment the rotation contribution is ΔT = + 0.29 μs.

The support ring and the pendulum bob also form a double pendulum, which changes the ideal period slightly. By the usual theory of small oscillations applied to the double pendulum,[15] the possible frequencies are given by

$$\omega^2 = \frac{1}{2}\left(1 + \frac{l_1}{l_2} + \frac{m_r d_r}{m l_1} + \frac{I_r}{m l_1 l_2}\right)\frac{m l_1 l_2}{I_r}\frac{g}{l_2}\left[1 \pm \left(1 - \frac{4(I_r / m l_1 l_2)[1 + m_r d_r / m l_1]}{[1 + (l_1/l_2) + (m_r d_r / m l_1) + (I_r / m l_1 l_2)]^2}\right)^{1/2}\right], \tag{14}$$

where $l_1 = d_r + r_0$ is the distance between the pivot and the edge of the ring and l_2 is the length of the pendulum wire. Taking the negative sign for $\omega \simeq \omega_0$ and carefully expanding the square root, we find that the correction to the period is

$$\frac{\Delta T}{T_0} = -\left(\frac{1}{2}\right)\left(\frac{m_r}{m}\right)\left(\frac{d_r}{l}\right) + \left(\frac{1}{2}\right)\frac{I_r}{ml^2} + \left(\frac{1}{2}\right)\left(\frac{m_r}{m}\right)^2\left(\frac{d_r^2}{l_1 l}\right), \tag{15}$$

where $l \equiv l_1 + l_2$. The first two terms represent the mass correction obtained above in Eq. (11). The third term is the double pendulum correction. This term implies ΔT = + 4.6 μs, a negligible amount. However, note that once again flexibility produces a positive correction, as in the rotation of the bob and in the bending of the wire itself.

C. Air corrections

Normally a pendulum experiment will take place in air. There are several ways in which the air changes the measured period.

1. Buoyancy

By Archimedes's principle the apparent weight of the bob is reduced by the weight of the displaced air. This property has the effect of increasing the period since the effective gravity is decreased. The correction is

$$\Delta T / T_0 = (\tfrac{1}{2})(m_a / m), \tag{16}$$

where m_a is the mass of the air displaced by the pendulum bob. The atmospheric pressure was 100.44 kPa (753.4 mmHg) and the temperature was 25.5 °C; the molecular weight of dry air is 0.02896 kg/mol. Thus the air density was 1.171 kg/m^3. The volume of the pendulum ball is 118.8 cm^3 (calculated) and the volume of the cap is 4.0 cm^3 (by displacement of water), so m_a = 0.1438 g. Therefore ΔT = + 292 μs.

2. Damping

The resistance of the air acts on both the pendulum ball and the pendulum wire. It causes the amplitude to decrease with time and increases the period of oscillation slightly. The law of force on any component of the system is determined by the Reynolds number for that component, defined as

$$R \equiv \rho VL/\eta\ , \tag{17}$$

where ρ and η are the fluid density and viscosity, V is a characteristic velocity, and L is a characteristic length. The drag force is usually expressed in the form

$$F = \tfrac{1}{2} C_D A \rho v^2\ , \tag{18}$$

where the drag coefficient C_D, is a dimensionless number which is a function of the Reynolds number. For values of R of the order 1 or less, the force is proportional to the velocity and C_D is proportional to R^{-1}. For values of R of the order 10^3 to 10^5 the force is proportional to the square of the velocity and C_D is a constant. In our experiment ρ = 1.171 kg/m^3 and η = 1.853 × 10^{-5} Pa ·s. The diameter of the wire was 0.320 ± 0.002 mm. The maximum Reynolds number based on diameter for the ball was 1100, where the quadratic force law should apply, while the maximum value based on diameter for the wire was 6, where the linear force law should prevail.

Since the damping force is neither linear nor quadratic, but rather a combination of the two, it makes sense to establish a damping function which contains both effects simultaneously. We compute the decrease in amplitude in a simple physical way using the work-energy theorem.[16] The work done by a damping force

$$F = b\,|\,v\,| + cv^2 \tag{19}$$

acting on the center of mass over the first half-period is

$$W = -\int_0^{\pi/\omega_0} Fv\ dt = -\int_0^{\pi/\omega_0} (-bv + cv^2) v\ dt\ . \tag{20}$$

Since W is much smaller than the pendulum energy we can use

$$\theta = \theta_0 \cos \omega_0 t \tag{21}$$

to compute the bob velocity and evaluate Eq. (20). The result is

$$W \simeq \tfrac{1}{2}\pi (b/\omega_0)(l\omega_0\theta_0)^2 + \tfrac{4}{3}(c/\omega_0)(l\omega_0\theta_0)^3\ . \tag{22}$$

By conservation of energy this work must equal the decrease in potential energy at the turning points

$$W = -\Delta PE \simeq -mgl\theta_0 \Delta\theta_0\ , \tag{23}$$

The change in amplitude $\Delta\theta_0$AO, occurs in time π/ω_0, so we obtain a differential equation

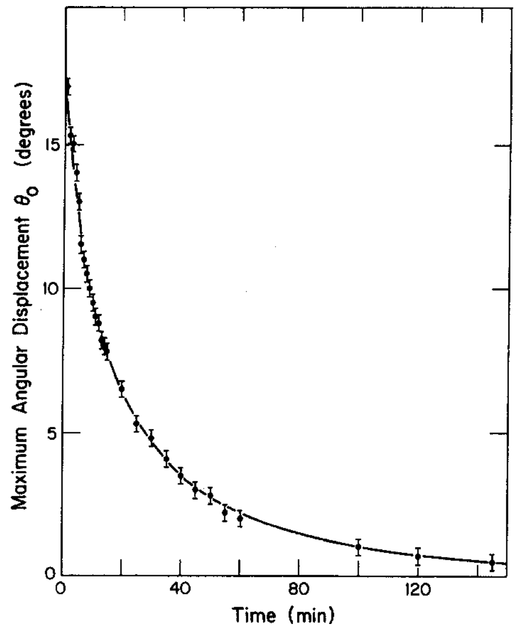

Fig. 2. Decay of amplitude with respect to time. The solid curve is given by Eq. (25) with parameters given by Eqs. (28) and (29). The standard error for a single measurement estimated from the residuals is 0.3° in agreement with the estimated uncertainty of the observations.

$$\frac{d\theta_0}{dt} = -\alpha\theta_0 - \beta\theta_0^2 \tag{24}$$

which is directly integrated to give

$$\theta_0 = \alpha\theta_{0m}\varepsilon^{-\alpha t} / [\beta\theta_{0m}(1 - e^{-\alpha t})\alpha]\ , \tag{25}$$

where $\theta_0 = \theta_{0m}$ at $t = 0$ and where the constants α and β are given by

$$\alpha = \tfrac{1}{2}(b/m) \tag{26}$$

and

$$\beta = \tfrac{4}{3}(\omega_0/\pi)(cl/m) \tag{27}$$

To test this damping formula we use the data shown in Fig. 2. The initial amplitude was θ_{0m} = 17.5 ± 0.3°. Using a nonlinear regression routine we vary the parameters α and β to optimize the fit of the data. As seen by the solid curve in Fig. 2, the fit is excellent and the fitted parameters are

$$\alpha = (2.49 \pm 0.11) \times 10^{-4}\,\mathrm{s}^{-1} \tag{28}$$

and

$$\beta = (3.17 \pm 0.03) \times 10^{-3}\,\mathrm{s}^{-1} \cdot \mathrm{rad}^{-1}\ . \tag{29}$$

Later in this article we will interpret these parameters in terms of the physical damping constants b and c of Eq. (19).

In order to determine the correction to the period we must consider the differential equation of motion. Since both damping forces are small we can take them as independent perturbations.[17]

For linear damping and small oscillations the equation of motion is

$$\ddot{\theta}+2\alpha\dot{\theta}+\omega_0^2\theta=0\,, \tag{30}$$

where $\alpha \equiv b/2m$. With initial conditions $\theta=\theta_{0m}$ and $\dot{\theta}=0$ at $t = 0$ the solution is[18]

$$\begin{aligned}\theta &= \theta_{0m}(\omega_0/\omega)e^{-\alpha t}\cos(\omega t-\delta)\\ &= \theta_{0m}e^{-\alpha t}[\cos\omega t+(\alpha/\omega)\sin\omega t]\end{aligned} \tag{31}$$

where $\cos\delta=\omega/\omega_0$ and

$$\omega=\omega_0[1-(\alpha/\omega_0)^2]^{1/2} \tag{32}$$

The damping term $2\alpha\dot{\theta}$ in Eq. (30) causes the amplitude to decrease as

$$\theta_0=\theta_{0m}e^{-\alpha t} \tag{33}$$

This result follows from Eq. (25) for $\beta=0$. There is also an increase in period given by

$$\Delta T/T_0=(\tfrac{1}{2})(\alpha/\omega_0)^2\,. \tag{34}$$

Using the experimental value of α we obtain $\Delta T=0.033\mu\text{s}$, which is negligible.

Next consider quadratic damping. Unlike the case of linear damping, an exact solution is not possible. An additional complicating feature is that the equation of motion is not even analytic since the sign of the force must be adjusted each half-period to correspond to a retarding force. The problem can be solved by means of a perturbation expansion using the method of Lindstedt and Poincaré[19, 20, 21, 22] as applied to the associated analytic problem where the sign of the force is not changed. In the constant sign case the first half-period is damped and the second half-period is negatively damped. The resulting motion is periodic so the Lindstedt–Poincaré method can be used. Only the first half-period corresponds to our damped pendulum problem, but the solution can be reapplied for subsequent half-periods.

The equation of motion corresponding to damped motion for the first half-period is

$$\ddot{\theta}-\varepsilon\dot{\theta}+\omega_0^2\theta=0\,, \tag{35}$$

where $\varepsilon\equiv cl/m$. We look for a periodic solution with period $T=2\pi/\omega_0$. The damping term introduces higher harmonics into the solution and also changes the period in a way that depends on the amplitude. We define a new independent variable

$$\phi\equiv\omega t \tag{36}$$

and convert Eq. (35) into the form

$$\omega^2\theta''-\varepsilon\omega^2\theta'^2+\omega_0^2\theta=0\,, \tag{37}$$

where $\theta'\equiv d\theta/d\phi$. We develop a perturbation expansion in terms of the small dimensionless parameter ε:

$$\theta=\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\cdots\,, \tag{38a}$$

$$\omega=\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots\,, \tag{38b}$$

The correction to the period is obtained from the requirement that the solution must be periodic, i.e., that it does not contain any secular terms which increase monotonically with time.

Substituting Eqs. (38) into Eq. (37), retaining terms through order ε^2, and setting the factors of each power of ε equal to zero (since the equation must hold for any value of ε, we obtain the following system of equations:

$$\psi_0''+\psi_0=0\,, \tag{39a}$$

$$2(\omega_1/\omega_0)\psi_0''+\psi_1''+\psi_1-\psi_0'^2=0\,, \tag{39b}$$

$$\begin{aligned}&[2(\omega_2/\omega_0)+(\omega_1/\omega_0)^2]\psi_0''+2(\omega_1/\omega_0)\psi_1''\\ &+\psi_2''+\psi_2\\ &-2(\omega_1/\omega_0)\psi_0'^2-2\psi_0'\psi_1'=0\end{aligned} \tag{39c}$$

We solve these equations recursively. The solution to Eq. (39a) such that $\psi_0=0$ and $\psi_0'=0$ at $\phi=0$ is

$$\psi_0=\theta_0\cos\phi\,. \tag{40}$$

Substituting Eq. (40) into Eq. (39b) and using the identity $\sin^2\phi=\frac{1}{2}-\frac{1}{2}\cos 2\phi$ we obtain

$$\begin{aligned}\psi_1''+\psi_1 &= 2(\omega_1/\omega_0)\theta_0\cos\phi\\ &+\tfrac{1}{2}\theta_0^2-\tfrac{1}{2}\theta_0^2\cos 2\phi\end{aligned} \tag{41}$$

The first term on the right introduces a term $(\omega_1/\omega_0)\theta_0\phi\sin\phi$ into the general solution. However, this latter term is a secular term, i.e., it does not satisfy the condition that the solution must be periodic. Therefore, we must have

$$\omega_1=0\,. \tag{42}$$

The solution to Eq. (41) that satisfies the initial conditions $\psi_1=0$ and $\psi_1'=0$ at $\phi=0$ is then

$$\psi_1=(\tfrac{1}{6})\theta_0^2(3-4\cos\phi+\cos 2\phi) \tag{43}$$

Hence at the end of the first half-cycle at $\phi=\pi$ the amplitude is

$$\theta_1=-\theta_0[1-(\tfrac{4}{3})\varepsilon\theta_0]\,. \tag{44}$$

Since the perturbation method assumes that the motion is periodic, it predicts that the pendulum would return to its original amplitude θ_0 at the end of the second half-oscillation. However, the only physically relevant solution corresponds to the interval $0\le t\le\pi/\omega$. During the next interval $\pi/\omega\le t\le 2\pi/\omega$ the damping force has the opposite sign and so Eq. (35) does not apply. If we make the transformation $\theta\to-\theta$ we can recover Eq. (35) and solve the problem for the new initial amplitude θ_1. Thus at the end of the second half-cycle the amplitude is

$$\theta_2=\theta_1[1-(\tfrac{4}{3})\varepsilon\theta_1]\,, \tag{45}$$

The amplitude gradually decreases as expected. Since the decrease in turning point $\Delta\theta=\theta_0-|\theta_1|$ occurs in time $\Delta t=\pi/\omega_0$, we can convert Eq. (44) into a differential equation which when integrated yields the amplitude as a function of time

$$\theta_0=(1+\beta\theta_{0m}t)^{-1}\theta_{0m}\,. \tag{46}$$

This result follows from Eq. (25) in the limit $\alpha \to 0$. Note that at large times the falloff is only inversely proportional to the time and not exponential as in Eq. (33), since the damping force falls off more rapidly as a function of velocity.

To obtain the correction to the period we must proceed to the second order and substitute Eqs. (40), (42), and (43) into Eq. (39c). The condition that there must not be any secular terms implies

$$\omega_2 = -(\tfrac{1}{6})\theta_0^2\omega_0 . \qquad (47)$$

Therefore, the correction to the period is

$$\Delta T / T_0 = (\tfrac{1}{6})\varepsilon^2\theta_0^2 = (3/32)(\pi\beta/\omega_0)^2\theta_0^2 . \qquad (48)$$

Using the experimental value of β we obtain $\Delta T = 0.027\mu\text{s}$, which is negligible.

In addition to having a small direct effect on the period, air resistance also has an indirect effect through the correction for finite amplitude. Suppose the amplitude decays from its initial value θ_{0m} to the value θ_{0f} during the time of measurement t_f. Then the period at some intermediate time is $T(t) = T_0[1 + \Delta T(t)/T_0]$ and the mean period as determined by counting oscillations is $\overline{T} = T_0(1 + \Delta\overline{T}/T_0)$, where the mean correction is given by

$$\frac{\Delta\overline{T}}{T_0} = \frac{1}{t_f}\int_0^{t_f}\frac{\Delta T(t)}{T_0}dt = \frac{1}{16t_f}\int_0^{t_f}[\theta_0(t)]^2\,dt . \qquad (49)$$

The total accumulation of phase is $\phi = 2\pi\int dt/T(t) \simeq 2\pi t_f/\overline{T}$ and the number of oscillations is $n = \phi/2\pi \simeq t_f/\overline{T}$. Substituting Eq. (25) into Eq. (49) and carrying out the integration, we obtain

$$\frac{\Delta\overline{T}}{T_0} = \frac{1}{16\beta t_f}\left[\left(1 - \frac{\theta_{0f}}{\theta_{0m}}\right)\theta_{0m} - \frac{\alpha}{\beta}\ln\frac{1+\beta\theta_{0m}/\alpha}{1+\beta\theta_{0f}/\alpha}\right], \qquad (50a)$$

where

$$t_f = (1/\alpha)\ln[(1+\alpha/\beta\theta_{0f})/(1+\alpha/\beta\theta_{0m})] . \qquad (50b)$$

By Eq. (25) $\theta_{of} = 0.869\theta_{0m}$ after 100 oscillations in a time of observation $t_f = 347.880$ s for an initial amplitude $\theta_{0m} = 3.0°$. Therefore, $\Delta\overline{T} = 519\mu\text{s}$. The difference caused by damping between this value and that calculated from Eq. (8) assuming constant amplitude is – 77 μs, which is not negligible.

If we take the limit, $\beta \to 0$ by expanding the logarithm in Eq. (50a) through terms of the second order, we obtain

$$\frac{\Delta\overline{T}}{T_0} = \frac{1}{32\alpha t_f}(\theta_{0m}^2 - \theta_{0f}^2) = \frac{\theta_{0m}^2 - \theta_{0f}^2}{32\ln(\theta_{0m}/\theta_{0f})}, \qquad (51)$$

in agreement with a formula derived by Jeffreys[3, 4] for linear damping. By Eq. (33) $\theta_{0f} = 0.917\theta_{0m}$ and so $\Delta\overline{T} = 547\mu\text{s}$. The effect caused by linear damping is thus – 49 us. If instead we take the limit $\alpha \to 0$ we obtain

$$\frac{\Delta\overline{T}}{T_0} = \frac{1}{16\beta t_f}(\theta_{0m} - \theta_{0f}) = \frac{1}{16}\theta_{0m}\theta_{0f} . \qquad (52)$$

By Eq. (46) $\theta_{0f} = 0.945\theta_{0m}$ and so $\Delta\overline{T} = 564\mu\text{s}$. The effect caused by quadratic damping is thus – 32 μs. The sum of the values for the two effects considered separately is nearly equal to the exact value when both effects are considered together.

3. Added mass

As the bob's motion varies during the pendulum cycle, the motion of the air surrounding the bob also varies. The kinetic energy of the system is thus partly that of the air. The effective mass of the system therefore exceeds the bob mass. The kinetic energy of the air can be taken into account by attributing an "added mass" m' to the bob's inertia (but not weight) proportional to the mass of the displaced air[1]:

$$m' = \kappa m_a , \qquad (53)$$

where κ is a constant. The correction to the period is

$$\Delta T / T_0 = (\tfrac{1}{2})(\kappa m_a / m) . \qquad (54)$$

The added mass accounts for the stirring of the air, which is part of the total system. Some air is also dragged along with the pendulum. In contrast, damping is a dissipative effect which results in the loss of energy to the system as a whole by conversion into heat. The need for the added mass correction was noted by Bessel in 1828.[1] Previously, it had been commonly thought that the only correction required owing to the air was that of buoyancy. (The added mass effect had been discovered independently by Du Buat in 1786, but it was not until after the appearance of Bessel's memoir that Du Buat's work attracted attention.) The dependence of added mass on viscosity was derived by Stokes in 1850.[1]

For steady motion in a perfect (nonviscous) fluid the kinetic energy of the air can be computed[23] from the velocity potential. The axially symmetric potential satisfying the conditions $v_r = 0$ at $r = a$ on the surface of the spherical bob and $\mathrm{v} = v_0\hat{z}$ at infinity is

$$\Phi = -v_0(r = a^3/2r^2)\cos\theta . \qquad (55)$$

Therefore, the velocity of the fluid is

$$\begin{aligned}\mathrm{v} = -\left(\frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}\right) &= v_0(\cos\theta\hat{r} - \sin\theta\hat{\theta}) \\ &- v_0\left(\frac{a}{r}\right)^3\left(\cos\theta\hat{r} + \tfrac{1}{2}\sin\theta\hat{\theta}\right)\end{aligned} . \qquad (56)$$

The first term represents a constant fluid flow since $\cos\theta\hat{r} - \sin\theta\hat{\theta} = \hat{z}$. In a reference frame at rest in the fluid

$$\mathrm{v} = -v_0(a/r)^3(\cos\theta\hat{r} + \tfrac{1}{2}\sin\theta\hat{\theta}) . \qquad (57)$$

The kinetic energy of the fluid flow is

$$\mathrm{KE} = \frac{1}{2}\rho\int_{\mathrm{v}} v^2 d\mathrm{V} = \frac{\pi}{3}\rho v_0^2 a^3 = \frac{1}{4} m_a v_0^2 . \qquad (58)$$

This is one-half the kinetic energy of the displaced fluid. Therefore, the result is that $\kappa = \frac{1}{2}$ and the effective mass of the pendulum is the bob mass plus one-half the mass of the displaced fluid.

In reality the value κ is greater than ½ because the motion of the bob is accelerated and the fluid has viscosity. Neglecting buoyancy, the force on the spherical bob due to the air may be expressed as[24, 25, 26, 27, 28]

$$\begin{aligned} -F &= \frac{1}{2} C_D \pi a^2 \rho \,|\, v \,|\, v + \frac{1}{2} C_A m_a \dot{v} \\ &+ 6 C_H a^2 (\pi\rho\eta)^{1/2} \int_{-\infty}^{t} (t-s)^{-1/2} \frac{dv}{ds} ds \end{aligned} . \qquad (59)$$

The first term is the drag force for steady motion given by Eq. (18) and the second term is the "added mass" contribution due to the kinetic energy imparted to the fluid. The third term involves the "history integral" which takes into account all previous motion; the force on the bob is partly determined by the effects of decaying vortices made at earlier times. The factors C_A, and C_H are empirical factors that depend on the relative importance of the convective acceleration compared to the local acceleration.

For oscillatory motion the history term in Eq. (59) becomes[1, 29] $C_H(3\pi a^2/\delta)(2\eta v + \delta^2 \rho \dot{v})$, where the penetration depth is defined as $\delta \equiv (2\eta/\omega_0\rho)^{1/2}$. Therefore, the equation of motion of the pendulum bob is

$$\begin{aligned} &\left[m + \left(\frac{1}{2} C_A + \frac{9}{4} C_H \frac{\delta}{a} \right) m_a \right] \dot{v} + 6 C_H \pi a^2 \frac{\eta}{\delta} v \\ &+ \frac{1}{2} C_D \pi a^2 \rho \,|\, v \,|\, v + (m - m_a) g \sin\theta = 0 \end{aligned} . \qquad (60)$$

Measurements on a sphere executing simple harmonic motion in a liquid indicate that the empirical factors C_A and C_H are given by[27, 28]

$$C_A = 2.10 - 0.132 M_A^2 / (1 + 0.12 M_A^2) , \qquad (61a)$$

$$C_H = 0.48 + 0.52 M_A^3 / (1 + M_A)^3 , \qquad (61b)$$

where $M_A \equiv | \dot{v} d / v^2 |$ is the acceleration modulus for a sphere of diameter *d*. The parameter C_A is evaluated at $\omega_0 t = (\frac{3}{4})\pi$, where the history term is zero, while the parameter C_H is evaluated at $\omega_0 t = (\frac{1}{4})\pi$, where the history term is maximum.[27] For a pendulum executing oscillatory motion the acceleration modulus for both parameters is thus $M_A = \sqrt{2} d / l\theta_0$. In our case M_A =0.55 so C_A = 2.06 and C_H = 0.50. In the limit $v \to 0$, $M_A \to \infty$ and $C_A = C_H = 1$.

The first term in Eq. (60) contains the added mass correction

$$m' = \kappa m_a = \left[\tfrac{1}{2} C_A + \tfrac{9}{4} C_H (\delta / a) \right] m_a . \qquad (62)$$

For C_A = 1 and δ = 0, κ= ½, as we obtained above for steady motion in a perfect fluid. However, with acceleration and viscosity taken into account we obtain κ = 1.18. The added mass *m'* is 0.02% of the bob mass. This theoretical result is in agreement with the classical pendulum experiments of the 19th century,[1, 30] The added mass correction to the period for our experiment by Eq. (54) is $\Delta T = +346\ \mu\mathrm{s}$. If the pendulum were a clock, it would lose 8.6 s in one day on account of added mass (and another 7.3 s due to buoyancy) compared to a similar pendulum swinging in a vacuum.[31]

4. Theoretical damping constants

The drag coefficient for a sphere is usually described by a graph derived from experimental data.[32] Many formulas have been proposed to represent this curve over various domains of the Reynolds number.[33] One convenient formula that has recently been suggested is[34]

$$C_D \simeq \frac{24}{R} + \frac{6}{1 + R^{1/2}} + 0.4 \equiv \frac{24}{R} + C'_D . \qquad (63)$$

Equation (63) is based on a fit of experimental data and is accurate to within 10% over the domain 0 < *R* < 2×10^5. At about $R \simeq 2.5\times10^5$ the boundary layer changes from laminar to turbulent and there is a sudden decrease in drag. The first term corresponds to the linear Stokes's law1 and the third term represents the Newtonian v^2 law. The second term accounts for the transition between these two limits. It is nearly constant over most of the oscillation and we shall evaluate it using the rms velocity. The sum of the last two terms C'_D is then 0.61. Thus by Eq. (60) we obtain

$$\begin{aligned} &(m + \kappa m_a)\dot{v} + 6\pi\eta a[1 + C_H(a/\delta)]v + \tfrac{1}{2} C'_D \pi a^2 \rho \,|\, v \,|\, v \\ &+ (m - m_a) g \sin\theta = 0 \end{aligned} . \qquad (64)$$

The linear drag constant *b* is 0.49×10^-4 kg/s and the quadratic drag constant *c* is 1.04×10^-3 kg/m.

The pendulum wire may be represented as a circular cylinder of infinite length. The drag coefficient[35] for Reynolds number 1 < R < 10 is approximately (empirically) given by

$$C_D \simeq (10/R) + 2 , \qquad (65)$$

where the Reynolds number is based on wire diameter. The added mass term[36] is negligible. No formulas are available for the history term for a cylinder but it, too, should be negligible since a fully developed wake does not form until $R \simeq 250$. The force per unit length on the wire is then

$$\frac{dF}{dx} = \frac{1}{2} C_D D \rho [v(x)]^2 , \qquad (66)$$

where *D* is the wire diameter, *x* is the distance of the wire element from the fulcrum, and $v(x) = x\ddot{\theta}$. The force acting on the pendulum center of mass equivalent to the total force acting on the wire is

$$F = \frac{1}{l}\int_0^l x\frac{dF}{dx}dx = \frac{5}{3}\eta l v + \frac{1}{4}\rho D l v^2 \ , \qquad (67)$$

where $v = l\dot{\theta}$. For the wire we obtain b = 0.93×10^{-4} kg/s and c = 0.28×10^{-3} kg/m. One interesting observation is that the linear drag constant is independent of the wire diameter. This is true as long as $C_D R$ is constant, but breaks down for very small Reynolds numbers (or wire diameters). For $R \ll 1$ the drag coefficient for a long cylinder is[37]

$$C_D = (8\pi / R)/[\tfrac{1}{2} - \gamma - \ln(R/8)] \, , \qquad (68)$$

where $\gamma \simeq 0.5772$ is Euler's constant. This result is the analog of Stokes's law for a sphere.

Another source of damping is friction at the point of support. Since the area of contact of the needle with the support is very small, the pressure is extreme. The adhesion between the two surfaces in contact will continually be formed and broken as the pendulum swings to and fro. In addition, the metal may suffer from fatigue and some motion will be communicated to the support. These effects will not materially influence the period, where they enter as corrections of the second order, but their contribution to the damping will not be negligible.

The experimental values of the drag constants from Eqs. (26) - (29) are $b = 2ma$ = (4.27 ± 0.19) × 10^{-4} kg/s and $c = (3/4)(\pi m/l\omega_0)\beta$ = (1.18 ± 0.01) × 10^{-3} kg/m. The calculated values of the total drag constants are b = 1.42 × 10^{-4} kg/s and c = 1.32 × 10^{-3} kg/m. The quadratic drag constants agree within 12%, but the calculated linear drag constant is only one-third the experimental value. The remainder must be attributed to damping by the suspension system. Stokes[1] found that the calculated rate of decrease of arc was about one-half the measured rate. However, when Stokes subtracted the logarithmic decrement for a pendulum in a vacuum from the logarithmic decrement for the same pendulum in air, the result was in almost exact agreement with the calculated value. Thus Stokes's analysis supports the interpretation of residual damping loss to the support system.

Since the drag coefficient of a sphere experimentally varies greatly depending on smoothness and other surface effects, it would be interesting to derive it from the measured data. If we subtract the contribution of the wire from the experimental quadratic drag constant, we obtain c = (0.90 ± 0.01) × 10^{-3} kg/m. The experimental drag coefficient for the sphere, corresponding to the v^2 drag law, is therefore $C'_D = 2c/\pi a^2\rho$ = 0.53 ± 0.01.

D. Elastic corrections

A real pendulum has neither an inextensible wire nor is it mounted on a perfectly rigid support. The latter effect is difficult to calculate, but can be essentially eliminated using a massive, rigid support. The string stretching effect can be estimated using Hooke's law.

1. String stretching

The length of the pendulum is increased by stretching of the wire due to the weight of the bob. The effective spring constant for a wire of rest length l_0 is

$$k = ES / l_0 \qquad (69)$$

where E is the elastic modulus (Young's modulus) and S is the cross-sectional area. For steel E = 2.0 × 10^{11} Pa. Thus by Hooke's law the increase in length is

$$\Delta l = mg / k = mgl_0 / ES \ . \qquad (70)$$

For our pendulum Δl = 1.6 mm, which is clearly not negligible. However, the length of the pendulum was measured while the pendulum was suspended in its static position. Thus this increment was automatically included in the total measurement.

There is also dynamic stretching of the wire from the apparent centrifugal and Coriolis forces acting on the bob during its motion. We can evaluate this effect by adapting a spring-pendulum system analysis[38] to the nearly stiff limit. Starting with the kinetic and potential energies defined by Eqs. (2) of Ref. 38, we modify the calculation for the nearly stiff spring case by changing from rectangular to polar coordinates:

$$x = l\sin\theta = z_0(1+\xi)\sin\theta \ , \qquad (71a)$$

$$z = z_0 - l\cos\theta = z_0[1-(1+\xi)\cos\theta] \, , \qquad (71b)$$

where $z_0 = l_0 + mg/k$ is the static pendulum length, $l = z_0(1 + \xi)$ is the dynamic length, ξ is the fractional string extension, and θ is the deflection angle. The equations of motion for small deflections are

$$(1+\xi)\ddot{\theta} + 2\dot{\theta}\xi + \omega_p^2\theta = 0 \, , \qquad (72a)$$

$$\ddot{\xi} + \omega_s^2\xi - \dot{\theta}^2 + \tfrac{1}{2}\omega_p^2\theta^2 = 0 \, , \qquad 72b$$

where $\omega_p \equiv (g/z_0)^{1/2}$ is the pendulum frequency and $\omega_s \equiv (g/m)^{1/2}$ is the spring (string) frequency. We look for a solution of the form

$$\theta = \theta_0 \cos\omega t \ , \qquad (73a)$$

$$\xi = a + b\cos 2\omega t \ , \qquad (73b)$$

since for a stationary mode the ξ (spring) motion must oscillate exactly twice as fast as the θ (pendulum) motion.[38] Substituting Eqs. (73) into Eq. (72a) and retaining only the first harmonic, we obtain

$$[\omega_p^2 - (1 + a - \tfrac{3}{2}b)\omega^2]\cos\omega t = 0 \ . \qquad (74)$$

Since Eq. (74) must hold for all t, the coefficient of cos ωt must be zero, so for small a and b,

$$\omega = \omega_p(1 - \tfrac{1}{2}a + \tfrac{3}{4}b) \, . \qquad (75)$$

Equation (72b) then becomes

$$\begin{aligned}&[\omega_s^2 a + \tfrac{1}{4}(\omega_p^2 - 2\omega^2)\theta_0^2] + [\omega_s^2 - 4\omega^2 - 4\omega^2)b \\ &+ \tfrac{1}{4}(\omega_p^2 + 2\omega^2)\theta_0^2]\cos 2\omega t = 0\end{aligned} \ . \qquad (76)$$

Both terms must vanish since the lowest neglected harmonic is the third. Since $\omega_s \gg \omega_p \simeq \omega$,

$$a = (\tfrac{1}{4})(\omega_p / \omega_s)^2 \theta_0^2 \,, \tag{77a}$$

$$b = -(\tfrac{3}{4})(\omega_p / \omega_s)^2 \theta_0^2 \,. \tag{77b}$$

The change in pendulum length thus fluctuates between $z_0\xi = -(\tfrac{1}{2})(mg/k)\theta_0^2 = -2.2\mu\text{m}$ at $\theta = \theta_0$ and $z_0\xi = +(mg/k)\theta_0^2 = +4.4\mu\text{m}$ at $\theta = \theta_0$. By Eq. (75) the resonant frequency is

$$\omega = \omega_p[1-(11/16)(\omega_p/\omega_s)^2\theta_0^2] \,. \tag{78}$$

This yields a period correction of

$$\frac{\Delta T}{T_0} = \left(\frac{11}{16}\right)\left(\frac{\omega_p}{\omega_s}\right)^2 \theta_0^2 = \left(\frac{11}{16}\right)\left(\frac{mg}{ES}\right)\theta_0^2 \,. \tag{79}$$

For our pendulum $\Delta T = 3.4\mu\text{s}$, which is negligible.

2. Support motion

To get a feeling for how rigid and massive the pendulum support must be, we model the support as a mass M kept in place by a spring of constant K, as shown in Fig. 3. The natural frequency of the support is thus

$$\Omega = (K/M)^{1/2} \,. \tag{80}$$

The coupled equations for the system of Fig. 3 are

$$\ddot{\theta} + \omega_0^2\theta + \ddot{x}/l = 0 \,, \tag{81a}$$

$$(1+m/M)\ddot{x} + (m/M)l\ddot{\theta} + \Omega^2 x = 0 \,. \tag{81b}$$

Equation (81a) implies that the effect of sway is to impart an additional angular acceleration $-\ddot{x}/l$ to the pendulum for small angles of oscillation. The frequency modes are found as usual by assuming both coordinates are proportional to $\cos\omega t$. Thus letting $\theta = \theta_0 \cos\omega t$ and $x = x_m \cos\omega t$ we obtain

$$(\omega^2 - \omega_0^2)\theta_0 + (\omega^2/l)x_m = 0 \,, \tag{82a}$$

$$(m/M)l\omega^2\theta_0 + [(1+m/M)\omega^2 - \Omega^2]x_m = 0 \,. \tag{82b}$$

For nontrivial solutions we must have

$$\begin{aligned} 2\omega^2 &= \Omega^2 + (1+m/M)\omega_0^2 \\ &\pm\{[\Omega^2 + (1+m/M)\omega_0^2]^2 - 4\omega_0^2\Omega^2\}^{1/2} \,. \end{aligned} \tag{83}$$

By Eq. (82a) the horizontal displacement of the support corresponding to frequency ω is

$$x_m = -[1-(\omega_0/\omega)^2]l\theta_0 \,. \tag{84}$$

The two limiting cases which follow are of interest to us.

a. Elastic rigidity: $\omega_0 \ll \Omega$ for any *m/M*. Choosing the negative sign in Eq. (83) for the lowest frequency mode and expanding the square root, keeping terms to order $(\omega_0/\Omega)^4$, we obtain

$$\omega^2 = \omega_0^2[1-(m/M)(\omega_0/\Omega)^2] \,. \tag{85}$$

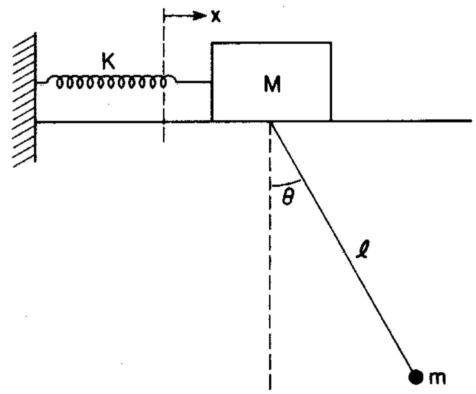

Fig. 3. Model of motion of the support by a system with two degrees of freedom.

The displacement of the support is

$$x_m = (m/M)(\omega_0/\Omega)^2 l\theta_0 = (mg/K)\theta_0 \tag{86}$$

and the period correction is

$$\frac{\Delta T}{T_0} = \left(\frac{1}{2}\right)\left(\frac{m}{M}\right)\left(\frac{\omega_0}{\Omega}\right)^2 = \left(\frac{1}{2}\right)\left(\frac{mg}{Kl}\right). \tag{87}$$

b. Inertial rigidity: $\Omega \ll \omega_0$. To get $\omega \simeq \omega_0$ we use the positive sign in Eq. (83) and the period correction is

$$\Delta T/T_0 = -(1/2)(m/M) \,. \tag{88}$$

By comparing Eqs. (87) and (88) we see that the support can either increase or decrease the period, depending on its nature. For both cases we should have a very massive support in order that the period be independent of the support. If four-figure accuracy is required, the support should be at least 10^4 times more massive than the pendulum bob or the natural support frequency should exceed the pendulum frequency by a factor of 100.

The assumption of elastic rigidity usually applies to most practical cases. In high precision pendulum measurements[3, 4] the constant K is either determined from interferometric observations of x_m or by means of a second pendulum of the same length and suspended from the same support which after time t acquires an amplitude $(\tfrac{1}{2})(x_m/l)\omega t$. Clark4 points out, however, that vibration will be communicated to the second pendulum by the surrounding air, as well as by the support, and so this effect must be corrected or eliminated.

We shall attempt an order of magnitude estimate of *K*. The pendulum support was a hook that was screwed securely into a wood beam in the ceiling of the laboratory.

The bending of the steel hook is negligible in comparison to the yielding of the wood surrounding the hook. The elastic constant for the wood is given approximately by

$$K = GA/d \,, \tag{89}$$

where G is the shear modulus, A is the projected area of the screw, and d is the thickness of the beam. In general, the shear modulus G is between one-third and one-half of the elastic modulus E, which for wood is typically 1.0×10^{10} Pa. Assuming A = 0.5 cm^2 and d = 5 cm, we obtain $K = 4\times10^6$ N/m. If we accept this estimate as reasonable, then x_m = 0. 1 μm and ΔT= 1 μs, which is negligible.

IV. CORRECTED PERIOD

The corrections to the ideal period and the corresponding changes in the computed acceleration of gravity are summarized in Table II. Since all the corrections are small we may assume that they add linearly. The net correction to the period is $\Delta T = +472\ \mu\text{s}$. Therefore, by Eqs. (4) and (7) the period of an equivalent ideal pendulum is

$$T_0 = 3.478\ 33 \pm 0.000\ 17\ \text{s} \tag{90}$$

and by Eq. (3) the measured acceleration due to gravity is

$$g = 9.8034 \pm 0.0014\ \text{m/s}^2\ . \tag{91}$$

Table II. Summary of pendulum corrections.

Effect	Correction to ideal period ΔT (μs)	Correction to gravitational acceleration Δg ($\times10^{-5}$ m/s^2)
Finite amplitude	+ 596.	+ 336.
Finite radius of bob	+ 72.	+ 41.
Mass of ring	- 282.	- 159.
Mass of cap	- 116.	- 65.
Mass of cap screw	+ 95.	+ 54.
Mass of wire	- 463.	- 261.
Flexibility of wire	+ 0.10	+ 0.06
Rotation of bob	+ 0.29	+ 0.16
Double pendulum	+ 4.6	+ 2.6
Buoyancy	+ 292.	+ 165.
Linear damping	+ 0.033	+ 0.019
Quadratic damping	+ 0.027	+ 0.015
Decay of finite amplitude	- 77.	- 43.
Added mass	+ 346.	+ 195.
Stretching of wire	+ 3.4	+ 1.9
Motion of support	+ 1.	+ 0.6
Total	+ 472.	+ 268.

This value is in agreement within 1 s.d. of the value 9.8012 ± 0.0035 m/s^2 obtained previously for the same site from the measurement of the acceleration of a freely falling body.[39] The latter value, however, did not take into account the apparent added mass of the falling body due to the motion of the air. If we represent the falling body as a "cigar-shaped" ellipsoid of length 13.4 cm and diameter 2.39 cm, the added mass is 0.050 times the mass of the displaced air.[40] The correction[41] to the measured acceleration is + 0.0003 m/s^2 and the resultant value of gravity is 9.8015 ± 0.0035 m/s^2.

The gravity value[42] at a geodesic reference station 550 m from the site is 9.802 531 5 ± 0.000 002 0 m/s^2 referenced to the International Gravity Standardization Net, 1971 gravity base network.[43] Taking into account slight corrections for the differences in elevation and latitude, we obtain 9.8026 m/s^2 as the actual value of g at the laboratory site.[39] The experimental values obtained by both methods agree with this geodesic value within 1 s.d.

V. DISCUSSION

The simple pendulum is a useful didactic system that one can repeatedly evaluate with success at virtually every level of physics. Besides using the pendulum as a gravimeter, one might test experimentally in greater detail some of the physical effects we have discussed here and use a variety of measurement techniques.[44] In the spirit of performing an experiment with relatively modest equipment that might be found in any student laboratory, we have measured the period with an ordinary stopwatch. However, it should be possible to improve the determination of g by another order of magnitude by measuring the period automatically with a photoelectric device and by measuring the length with a cathetometer. For a 1-m pendulum it would be necessary to have a timing circuit that could measure the period with a precision of 10 μs, or 1 ms for 100 oscillations. Cathetometers are available commercially[45] which have two traveling telescopes mounted on a vertical bar whose scale can be read with a vernier to 0.01 mm over a range of 100 cm. The effects of the air might be studied by using a smaller pendulum suspended within an evacuated Bell jar. With respect to damping, there seems to be no advantage to replacing the usual pendulum knife edge with a needle. However, the manufacturer intended our apparatus to be used also as a Foucault pendulum and thus provided a rotational degree of freedom.

If one wishes to push the experiment to six-figure accuracy, as in past actual measurements in standards laboratories, rather heroic efforts must be attempted. In the measurement of gravity at the National Bureau of Standards in Washington in 1935, Heyl and Cook[3, 4] used a reversible pendulum in the form of a uniform rod of fused silica. The pendulum was provided with two planes, upon each of which it could be swung in turn about a knife edge mounted in a steel support. The equivalent period of an ideal pendulum is expressed in terms of the two times of swing (adjusted to be nearly equal) and the distance between the planes. The apparatus was operated in a vacuum within a sealed case. The observation room was temperature controlled. In addition to the effects we have considered here, it was necessary to take into account clock errors, variation in temperature, change in length due to atmospheric pressure, imperfections of the knife edge, and alignment errors.

The level of sophistication of the experimental apparatus determines the precision of a measurement. The *accuracy* of the measurement, however, depends on the analysis of all the incidental factors that attend the experiment. Within those incidental factors lies a rich variety of physics.

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[1] The study of the motion of a pendulum in a viscous fluid has a long history. Fundamental contributions were made by F. W. Bessel, "Untersuchungen iiber die Länge des einfachen Secundenpendels," Abh. Berlin Akad. Aus dem Jahre 1826 (issued 1828; reprinted by H. Bruns, Leipzig, 1889). The classic paper on the subject is by G. G. Stokes, Trans. Camb. Philos. Soc. **9**, 8 (1851); *Mathematical and Physical Papers* (Cambridge U.P. London, 1922), Vol. 3, p. 1.
[2] A collection of pendulum corrections and their original references is found in the review article of F. Auerbach, in *Handbuch der Physikalischen und Technischen Mechanik,* edited by F. Auerbach and W. Host (Verlag von Johann Ambrosius Barth, Leipzig, 1930), Vol. IL
[3] An historical review of early pendulum measurements is given in J. H. Poynting and J. J. Thomson, A *Text-book of Physics, Volume I: Properties of Matter* (Griffin, London, 1920), 8th ed., Chap. 2. An important absolute measurement of g in Potsdam, Germany that remained the base of the international gravity network until 1971 was made by F. Kühnen and P. Furtwängler, "Bestimmung der absoluten Grösse der Schwerkraft zu Potsdam mit Reversionspendeln" (Veröff. K. Preuss. Geod. Inst. N.F. n. 7, 1906). For a review and discussion of modern absolute gravity experiments, see A. H. Cook, Metrologia **1**, 84 (1965).
[4] An example of a precision pendulum measurement is provided by the determination of gravity at the National Bureau of Standards in Washington with a reversible pendulum by P. R. Heyl and G. S. Cook, J. Res. Natl. Bur. Stand. **17**, 805 (1936). A reversible pendulum was also used to determine the value of gravity at the National Physical Laboratory in Teddington, England by J. S. Clark, Philos. Trans. R. Soc. London Ser. A **238**, 65 (1940). The calculations for both experiments were revised by H. Jeffreys, Mon. Not. R. Astron. Soc. London, Geophys. Suppl. **5**, 398 (1949).
[5] The most accurate method for the measurement of *g* is now by direct measurement of the acceleration of a freely falling body. See J. E. Faller, Ph.D. thesis, Princeton University, 1963 (University Microfilms, Ann Arbor, MI); D. R. Tate, J. Res. Natl. Bur. Stand. **72C**, 1 (1968).
[6] Distributed by Lapine Scientific Company, Chicago, IL, cat. no. Y34639.
[7] E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge U.P., New York, 1937), 4th ed., p. 73.
[8] H. Lamb, *Dynamics* (Cambridge U.P., New York, 1923), 2nd ed., p. 107.
[9] L. P. Fulcher and B. F. Davis, Am. J. Phys. **44**, 51 (1976); C. Kittel, W. D. Knight, and M. A. Ruderman, *Mechanics: Berkeley Physics Course, Vol. 1* (McGraw-Hill, New York, 1965), pp. 197-199.
[10] S. C. Zilio, Am. J. Phys. **50**, 450 (1982).
[11] S. T. Epstein and M. G. Olsson, Am. J. Phys. **45**, 671 (1977).
[12] C. G. Montgomery, Am. J. Phys. **46**, 411 (1978).
[13] J. A. Cronin, D. F. Greenberg, and V. L. Telegdi, *University of Chicago Graduate Problems in Physics* (Addison-Wesley, Reading, MA, 1967), p. 99.
[14] J. V. Hughes, Am. J. Phys. **21**, 47 (1953). The need for this correction was first pointed out by Laplace. See Bessel, Ref 1, p. 146; Poynting and Thomson, Ref. 3, p. 13.
[15] S. Timoshenko, *Vibration Problems in Engineering* (Van Nostrand, New York, 1955), 3rd ed., p. 196; K. Rossberg, *A First Course in Analytical Mechanics* (Wiley, New York, 1983), p. 246.
[16] H. M. Hansen and P. F. Chenea, *Mechanics of Vibration* (Wiley, New York,1952), p.322.
[17] The effects of finite amplitude, linear damping, and quadratic damping are considered as simultaneous perturbations by E. J. Routh, *Dynamics of a System of Rigid Bodies* (Macmillan, London, 1905; reprinted by Dover, New York, 1955), Part II, 6th ed., pp. 255-257.
[18] K. R. Symon, *Mechanics* (Addison-Wesley, Reading, MA, 1971), 3rd ed., p. 47.
[19] R. Bellman, *Perturbation Techniques in Mathematics, Physics, and Engineering* (Holt, Rinehart and Winston, New York, 1966; reprinted by Dover, New York, 1972), p. 57.
[20] L. A. Pipes and L. R. Harvill, *Applied Mathematics for Engineers and Physicists* (McGraw-Hill, New York, 1970), 3rd ed., p. 614.
[21] N. Minorsky, *Nonlinear Oscillations* (Van Nostrand, Princeton, NJ, 1962); N. Minorsky, *Introduction to Nonlinear Mechanics* (Edwards, Ann Arbor, MI, 1947); J. J. Stoker, *Nonlinear Vibrations* (Interscience, New York, 1950), p. 59.
[22] L. Meirovitch, *Elements of Vibration Analysis* (McGraw-Hill, New York, 1975), p. 364.
[23] H. Lamb, *Hydrodynamics* (Cambridge U.P., New York, 1932; reprinted by Dover, New York, 1945), 6th ed., p. 124.
[24] A. B. Basset, *A Treatise on Hydrodynamics* (Deighton, Bell, London, 1888; reprinted by Dover, New York, 1961), Vol. II, Chap. 22.
[25] L W. Strutt, Lord Rayleigh, Philos. Mag. **21**, 697 (1911); *Scientific Papers* (Cambridge U.P., London, 1920; reprinted by Dover, New York, 1964), Vol. VI, p. 29.
[26] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, MA, 1959), p. 97.
[27] F. Odar and W. S. Hamilton, J. Fluid Mech. **18**, 302 (1964).
[28] R Cliff, J. R. Grace, and M. E. Weber, *Bubbles, Drops, and Particles* (Academic, New York, 1978), p. 296.
[29] Reference 23, p.644; Reff. 24, p.267; Ref. 26, p. 96; Ref. 28, p. 287.
[30] G. Birkhoff, *Hydrodynamics* (Princeton U.P., Princeton, NJ, 1960), rev. ed., p. 150.
[31] For the same reason a mechanical wristwatch would run faster at a high altitude, where the air is less dense, than at sea level. See E. J. Routh, *Dynamics of a System of Rigid Bodies* (Macmillan, London, 1905; reprinted by Dover, New York, 1960), Part 1, 7th rev. ed., p. 82; see also Ref. 13, p. 100.
[32] A wealth of experimental data may be found in S. F. Hoerner, *Fluid Dynamic Drag* (Hoerner, Brick Town, NJ, 1965).
[33] Reference 28, p. 111.
[34] R M. White, *Viscous Fluid Flow* (McGraw-Hill, New York, 1974), p. 209.
[35] Reference 28, p. 155.
[36] Reference 23, p. 77.
[37] Reference 23, p. 616; Ref. 26, p. 68.
[38] M. G. Olsson, Am. J. Phys. **44**, 1211 (1976).
[39] R. A. Nelson, Am. J. Phys. **49**, 829 (1981).
[40] Reference 23, p. 154.
[41] In Ref. 39 it was stated incorrectly that the effect of the added mass would be to imply a negative correction.
[42] G. P. Woolard, L. F. Machesky, and R. L. Longfield, *Results for a Series of North-South Gravity Control Traverses in the United States* (Hawaii Institute of Geophysics, Univ. of Hawaii, Honolulu, 1969), Part 3, Final Report [HIG-69-22], station ref. no. NS1354; *DMAAC Reference Publication #25, Part 2.. World Relative Gravity Reference Network for North America* (Defense Mapping Agency, Aerospace Center, St. Louis, MO, 1970; revised 1974), station ref. no. DoD 0438-1.
[43] C. Morelli, Ed., *International Gravity Standardization Net 1971,* Spec. Publ. No. 4 (International Association of Geodesy, Paris, 1974).
[44] R. C. Nicklin and J. B. Rafert, Am. J. Phys. **52,** 632 (1984) and references cited therein.
[45] The Ealing Corporation, South Natick, MA, cathetometer/comparator, cat. no. 11-5329.