Attempted answers and copies:

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NOT JUST THE USUAL COUPLED OSCILLATORS, and a diversion into two usual ones.

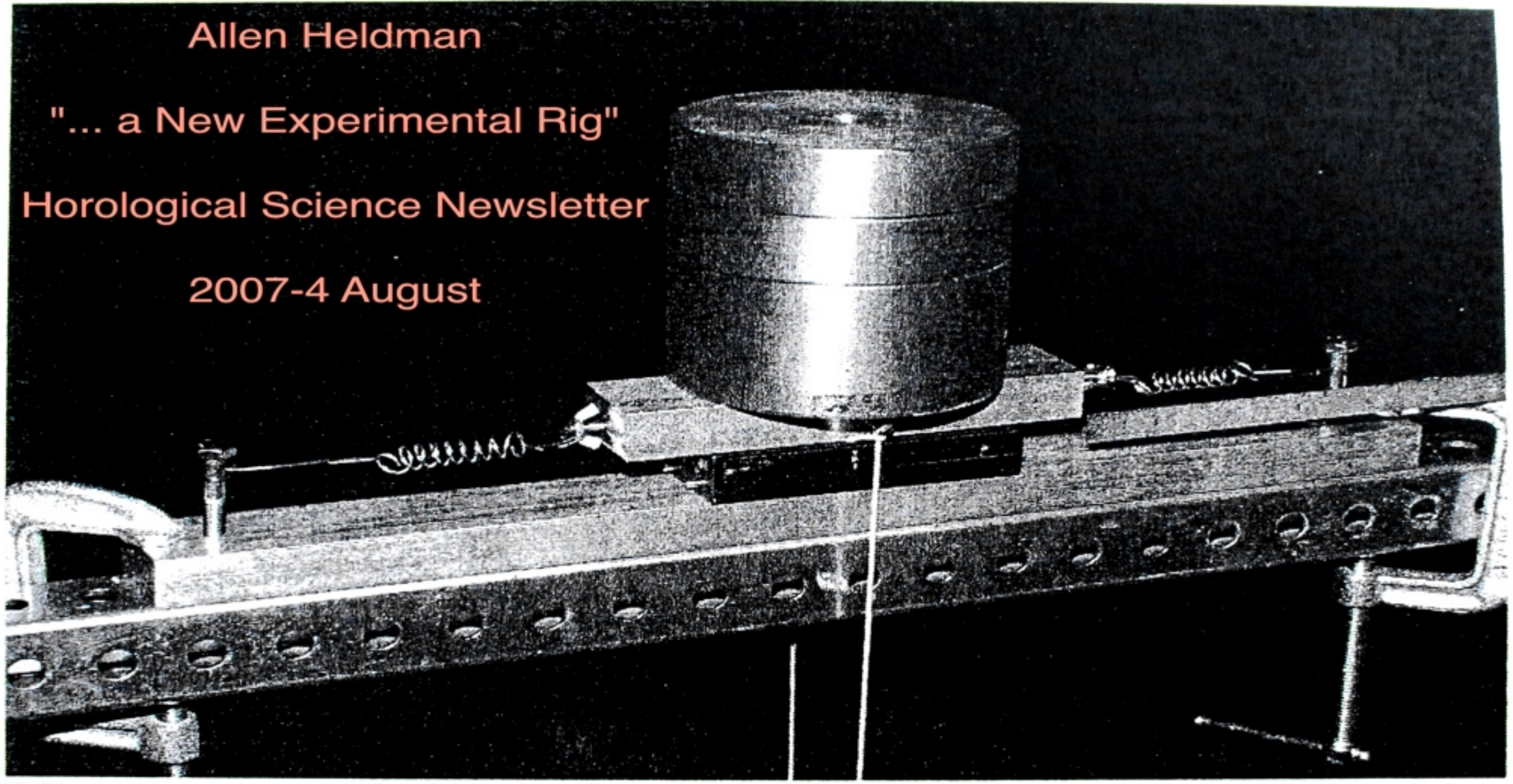
The Origin of My Study:

Questions from an horological equipment manufacturer friend piqued my interest in pendula. One question was originally from one of his amateur horologist friends who sent me the first part of his article in the Horological Science Newsletter, in which he initiated a discussion on pendulum dissipation due to support compliance.

Pendulum Support Loss Measured With a View to Optimal Design and Mounting - Part 1

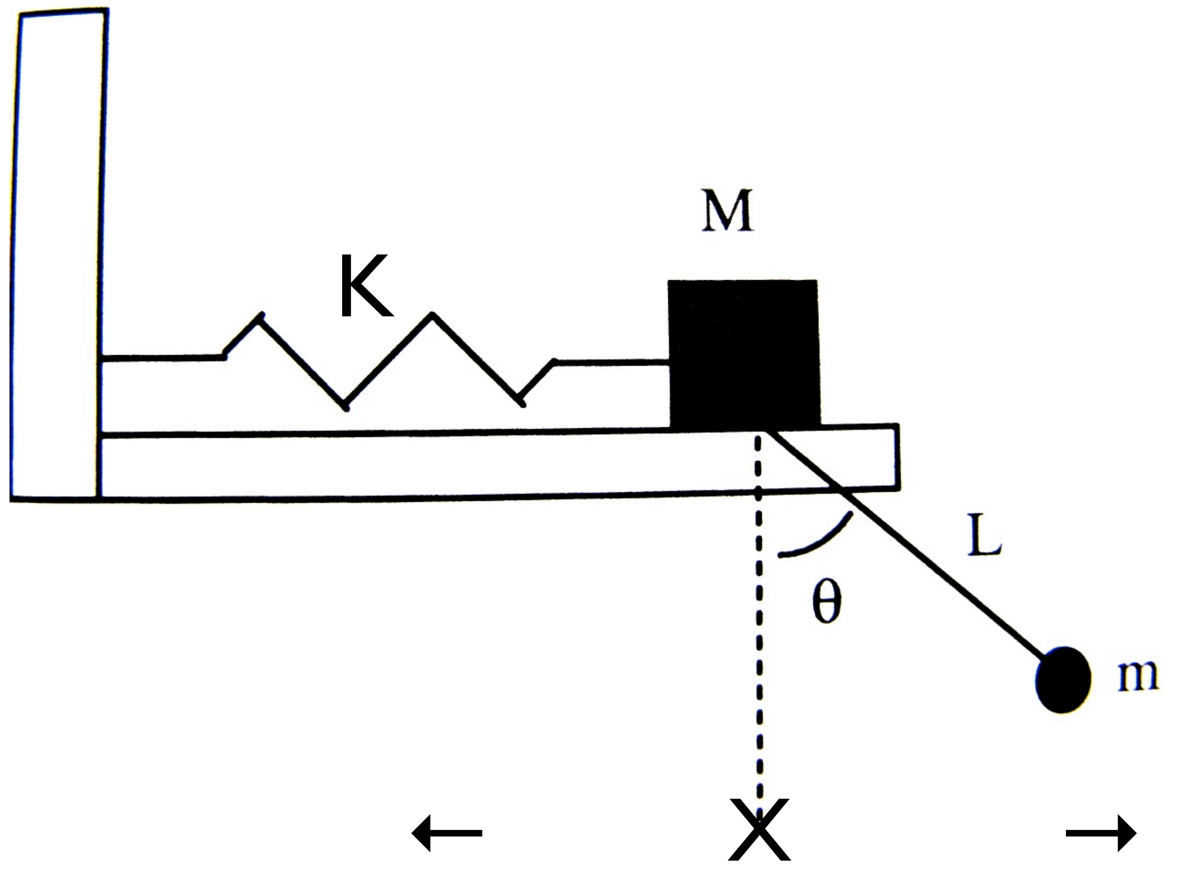
Horological Science Newsletter, 2006-5   December 2006, by Alan W. Heldman

He measured the pendulum amplitudes of a Synchronome clock mounted on a load bearing wall and in a free standing case with varying bob masses. He found significant loss due to suspension movement, which he determined from the pendulum's "steady state" amplitudes. He then measured statically with a force gauge, and dynamically with the pendulum oscillating, the compliance of the two supports. Curiously he found a discrepancy of about 2.5X. He concluded that a "sort of resonance" was occurring. This prompted him to construct the following apparatus to learn about "resonance" and perhaps model what was taking place.



I need to explain the rig. The two white vertical (low-stretch Spectra) cords are a bifilar pendulum "rod" for a bob. The cords hang from the front side and back side of that shorter shiny horizontal bar (aluminum, 0.5" x 1.5" x 6") which has the cylindrical weights on it and which has the taut helical springs at its ends. That bar is free to slide back and forth, except as restrained by the springs, because that black thing under it is a high-grade "ball-slide", rated for 24 pounds load. It allows the bar above it to slide back and forth with very low friction, even when a lot of weight is on the bar or is hanging from the bar, or both.

Beyond describing the apparatus and manually driving the spring oscillator, he didn't further experiment. So, intrigued and not having a similar apparatus, I numerically simulated a pendulum whose support is a spring oscillator. After laboriously finding the Lagrangian and plugging into the formula, I, fortunately, because of my errors, found Cooper and Peregrine in *Modern Analytic Mechanics, p. 117* had done this.



$$(M+m)\ddot{X}/L + m(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + KX/L = 0$$

$$m(\ddot{\theta} + \frac{\ddot{X}}{L}\cos\theta) + m\frac{g}{L}\sin\theta$$

Both equations divided by L^2.

I then separated the accelerations with the following result:

$$Q_1: \; d^2X/dt^2 = [L\mu P\sin\theta\cos\theta + L\mu(d\theta/dt)^2 - SX]/[1+\mu\sin\theta^2], \; \textit{and}$$

$$Q_2: \; d^2\theta/dt^2 = [-\mu(d\theta/dt)^2\sin\theta\cos\theta/(1+\mu) + SX\cos\theta/L(1+\mu) - P\sin\theta]/$$
$$[1-\mu\cos^2\theta/(1+\mu)$$

Where:

$X =$ cart's position, $t =$ time, $L =$ rod length, $\mu = m/M$ (bob and support masses), $P =$ g/L,
$\theta =$ rod angle wrt vertical, and $S = K/M$ ($K =$ spring's force constant)

Note: S^2 is the spring oscillator's natural frequency, and P^2 is the pendulum's. Also the coupling is a positive function of μ.

The following is my most recent True Basic program. It's a "leapfrog integration algorithm". The method is described in detail by Eisberg and Lerner (Physics/ Foundations and applications) and earlier by Feynman et alii (*The Feynman* Lectures on Physics). See also:

http://en.wikipedia.org/wiki/Leapfrog_integration

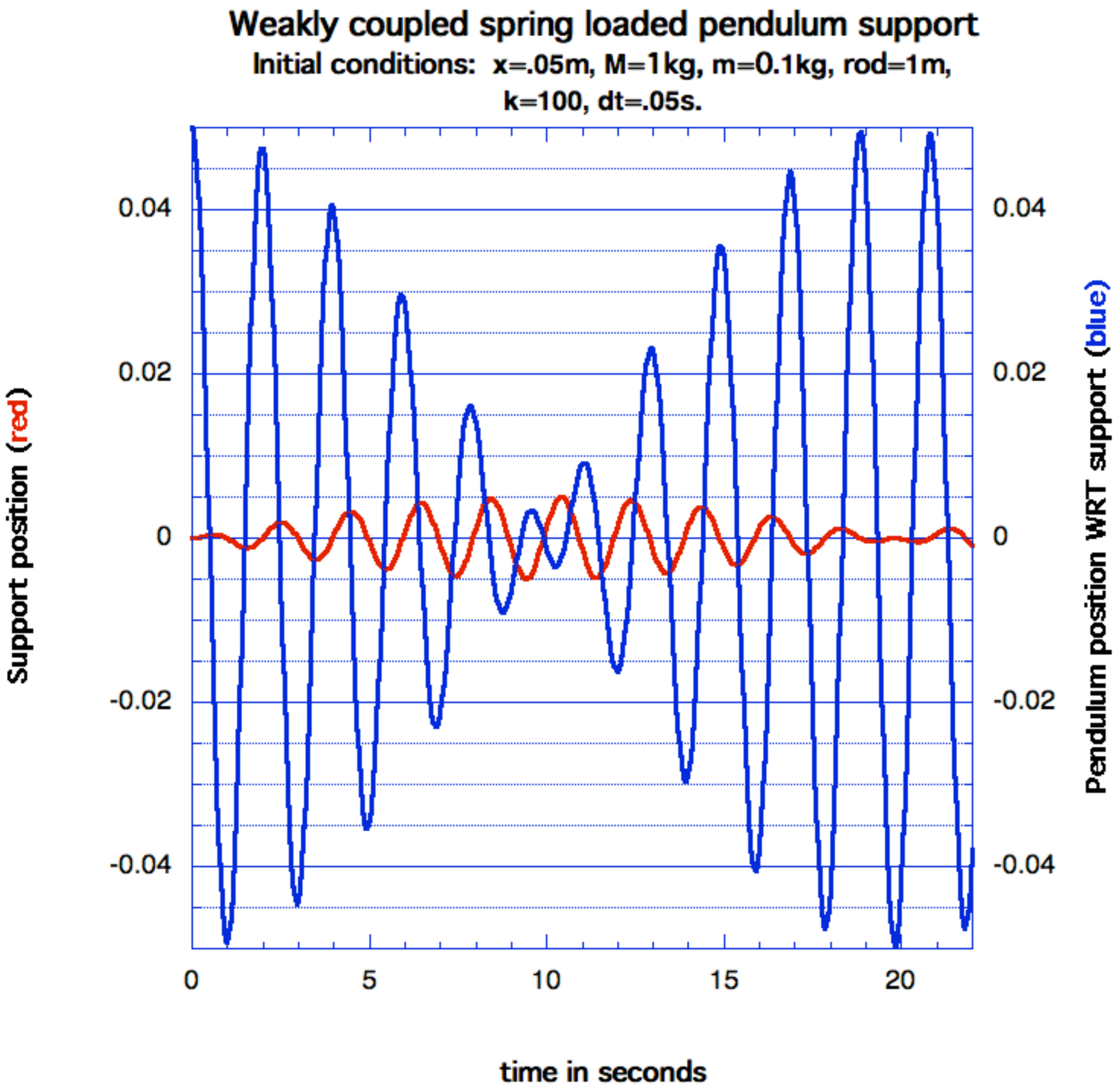
Warning: not tested.

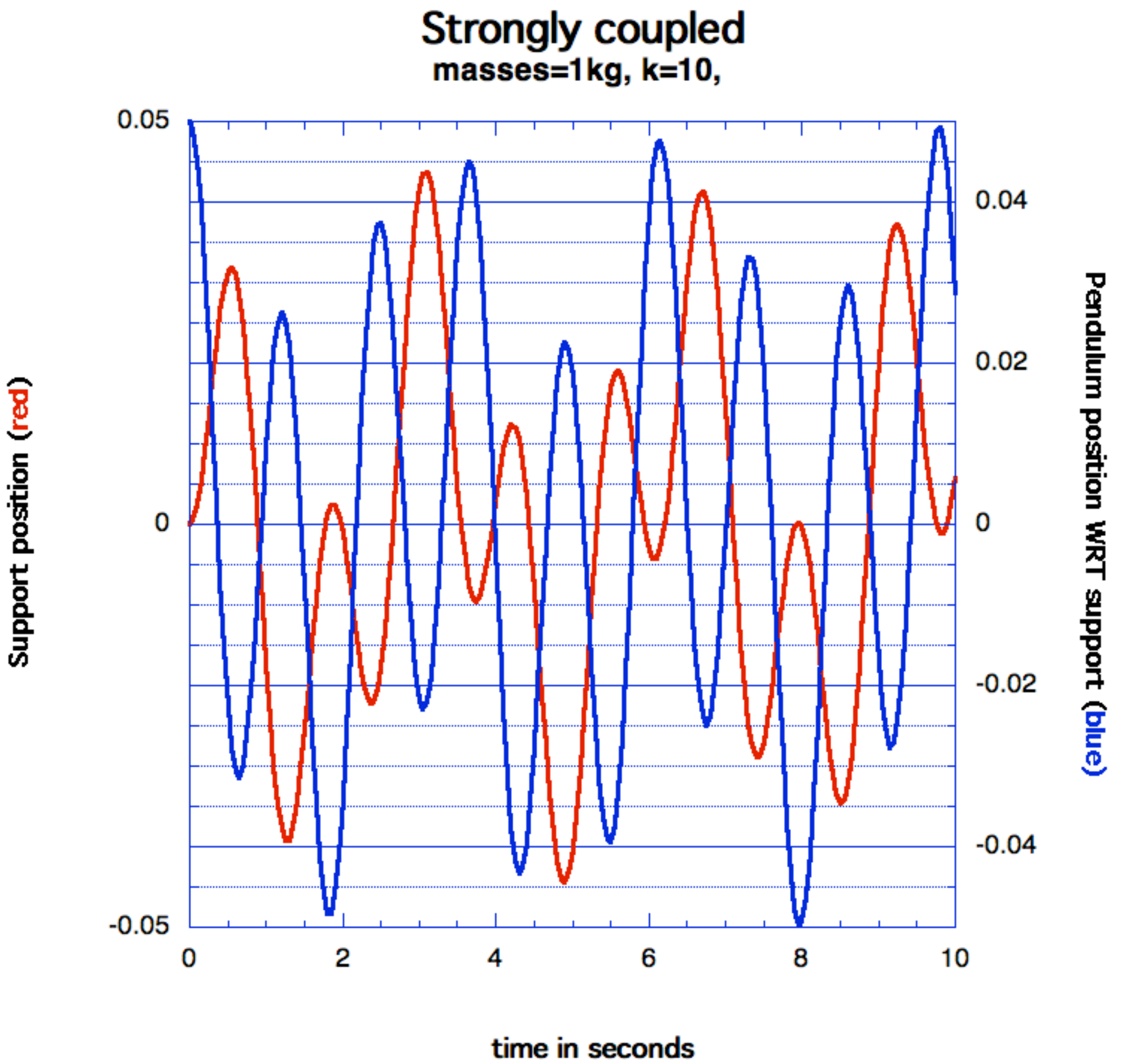
```
!  Cooper and Pellegrini.txt  [exact] w/ viscous damping. This program finds position (X1 and theta) as a function of time and initial conditions.
!  Parameters:  delta t, initial positions, masses, rod length, spring constant, g= 10 N/m, etc.).
PRINT "this is the first line and g = 9.8"
PRINT "input X1 (initial M1 position)"
INPUT X1
LET V1=0
PRINT "input X2 (initial pendulum displacement in radians)"
INPUT X2
LET V2= 0
LET t=0
PRINT "input delta t. Suggest 0.001 seconds"
INPUT D
PRINT "input M1 & M2 (kg)  Masses, spring and bob"
INPUT M1
INPUT M2
PRINT "input l (meters) & k (N/m); rod length and spring constant"
INPUT l
INPUT k
LET g = 9.8
print "input viscous dissipation constant"
input vdc
PRINT "number of steps [total time printed is number of steps times delta t] "
INPUT stop
PRINT "input mod number m, i.e. mod (steps, m).  Print interval (secs./print) is delta t times mod number (m)"
INPUT m
LET mu = M2/M1
LET S=k/M1
print "Wos=sqr(k/m); T0s = 2Pi sqrt(M1/K)"
print "Ts = "
print 2*pi*sqr(1/S)                    ! natural spring frequency (radians / sec)^2
LET P=g/l
print " W0p =sqr(g/L)"
print "Tp = "
print 2*Pi*sqr(l/g)
```

```
SUB calc
   LET Q1 = ( (L*mu*P*cos(X2)*sin(X2) + L*mu*V2*V2*sin(X2) - S*X1) / ( 1+mu-mu*cos(X2)*cos(X2) ) ) - v1 * vdc/m1


   LET Q2 = ( ( ( S*(X1/l)*cos(X2) )/(1+mu) ) - ( (mu*sin(X2)*cos(X2)*V2*V2) / (1+mu) ) - P*sin(X2) ) / ( 1- ( mu*cos(X2)*cos(X2) )/(1+mu) )


END SUB
SUB type
   IF mod (steps, m) = 0 then PRINT t, X1, X2
END SUB

CALL calc
LET V1 = V1 + Q1 * D / 2          ! increased accuracy when calculated in the middle of the interval.
LET V2 = V2 + Q2 * D / 2

FOR steps = 1 to stop
   IF steps = 1 then PRINT t,  X1, X2
   CALL type
   LET X1 = X1 + V1 * D
   LET X2 = X2 + v2 * D
   LET T = T + D
   CALL calc

   LET V1 = V1 + Q1 * D
   LET V2 = V2 + Q2 * D

NEXT steps
END
```

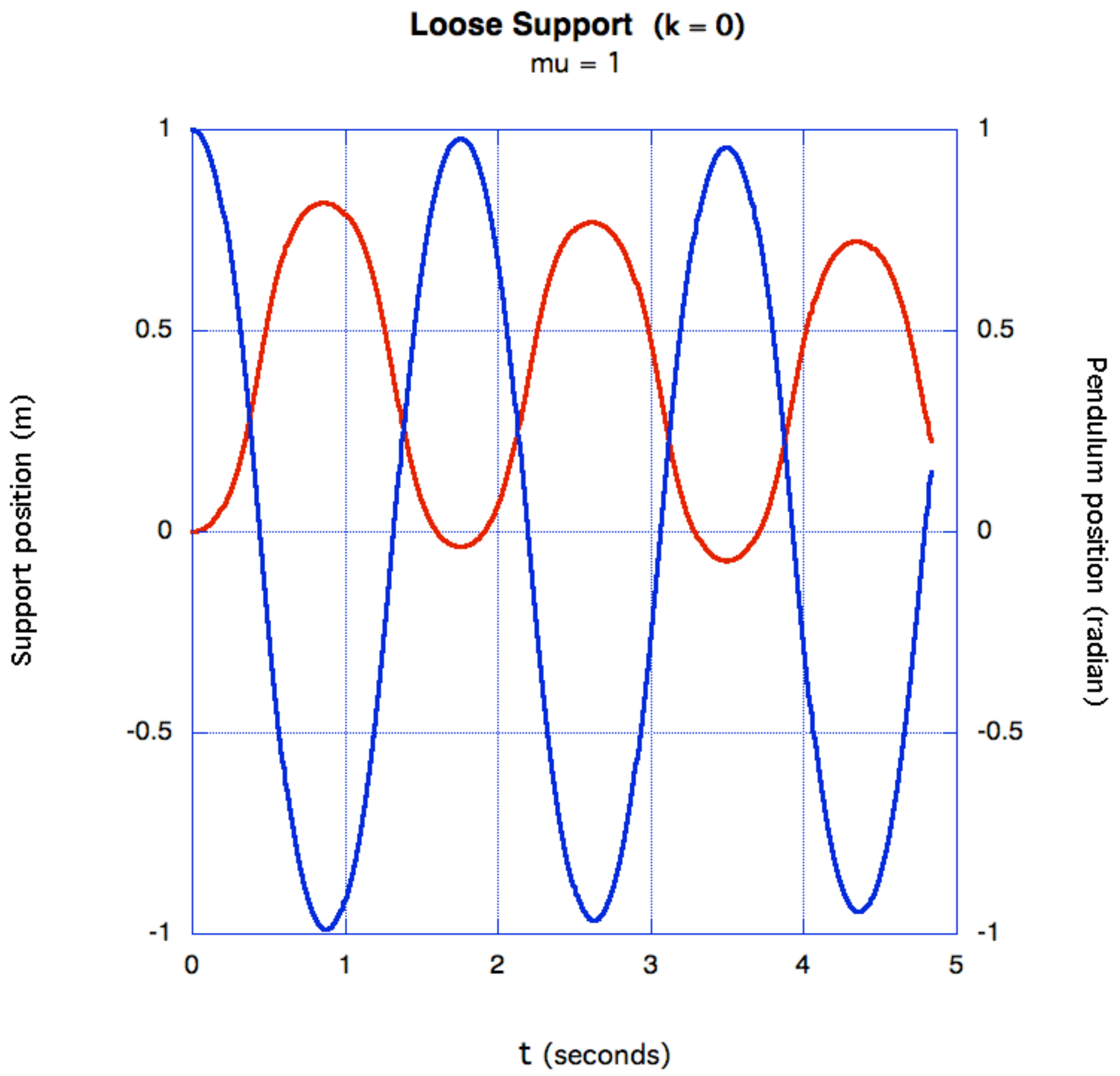
Here's the output of an earlier version without damping.

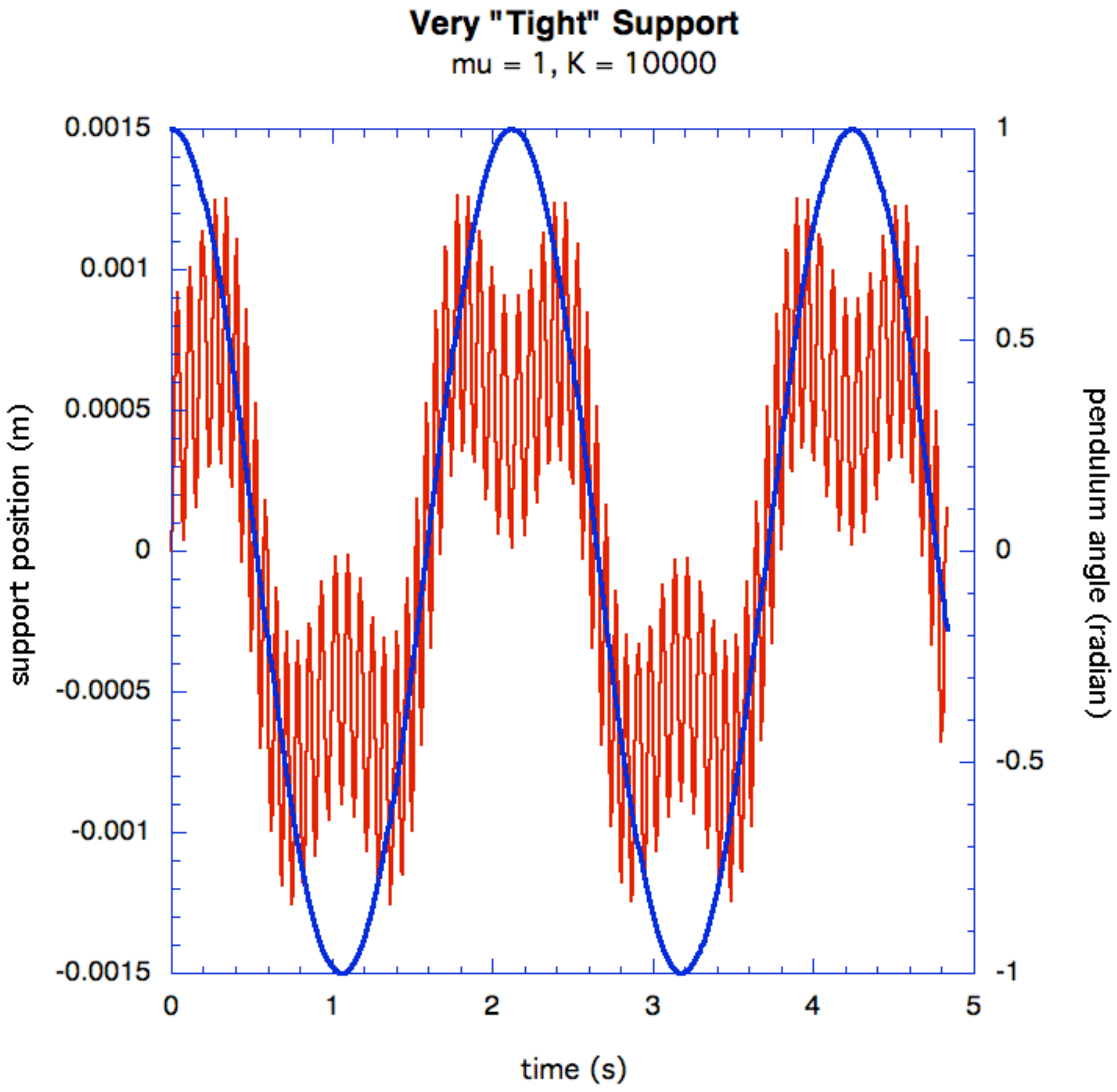
Units are meter and radian.



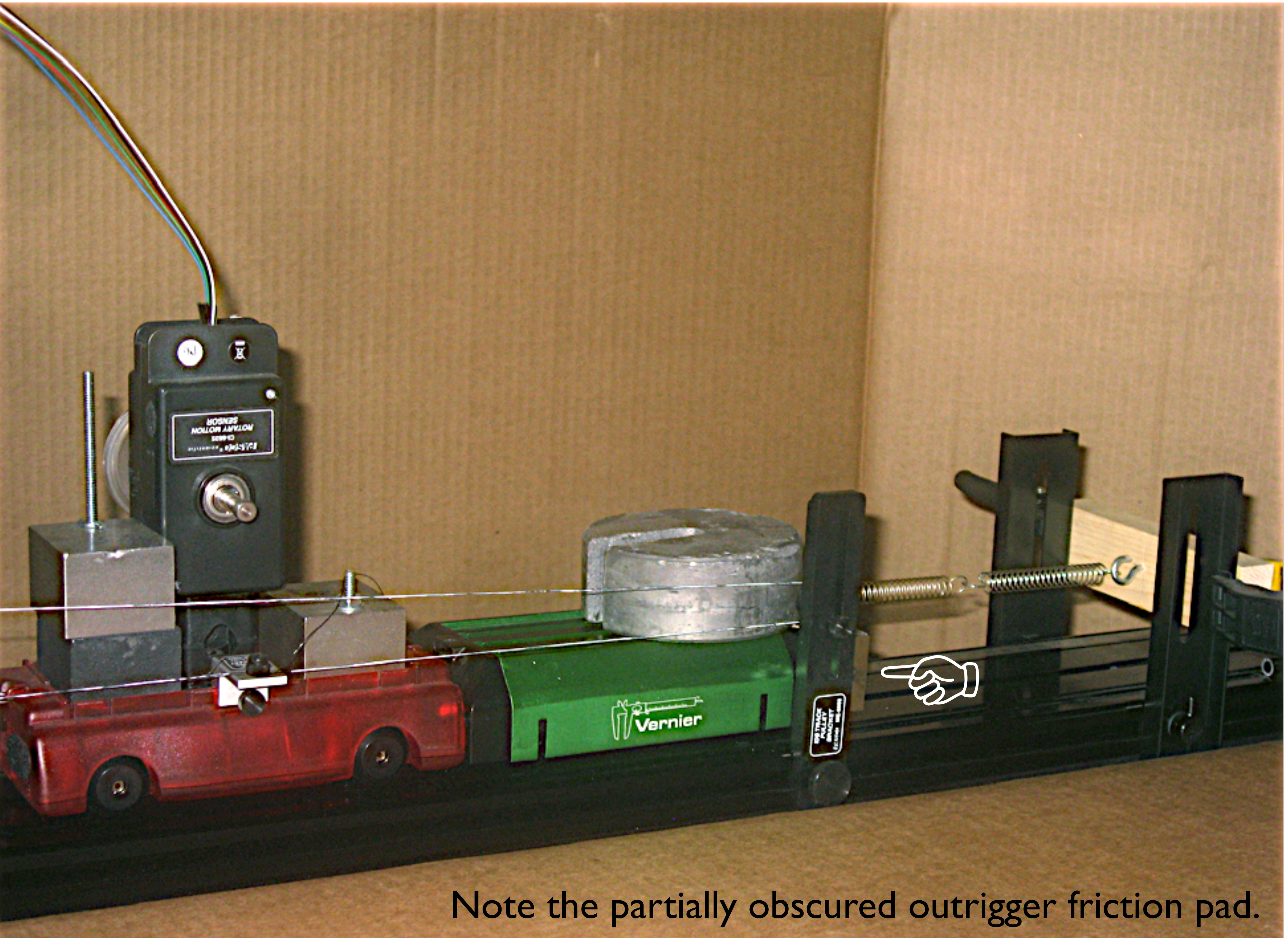


These compare free (no spring) and stiff supports.

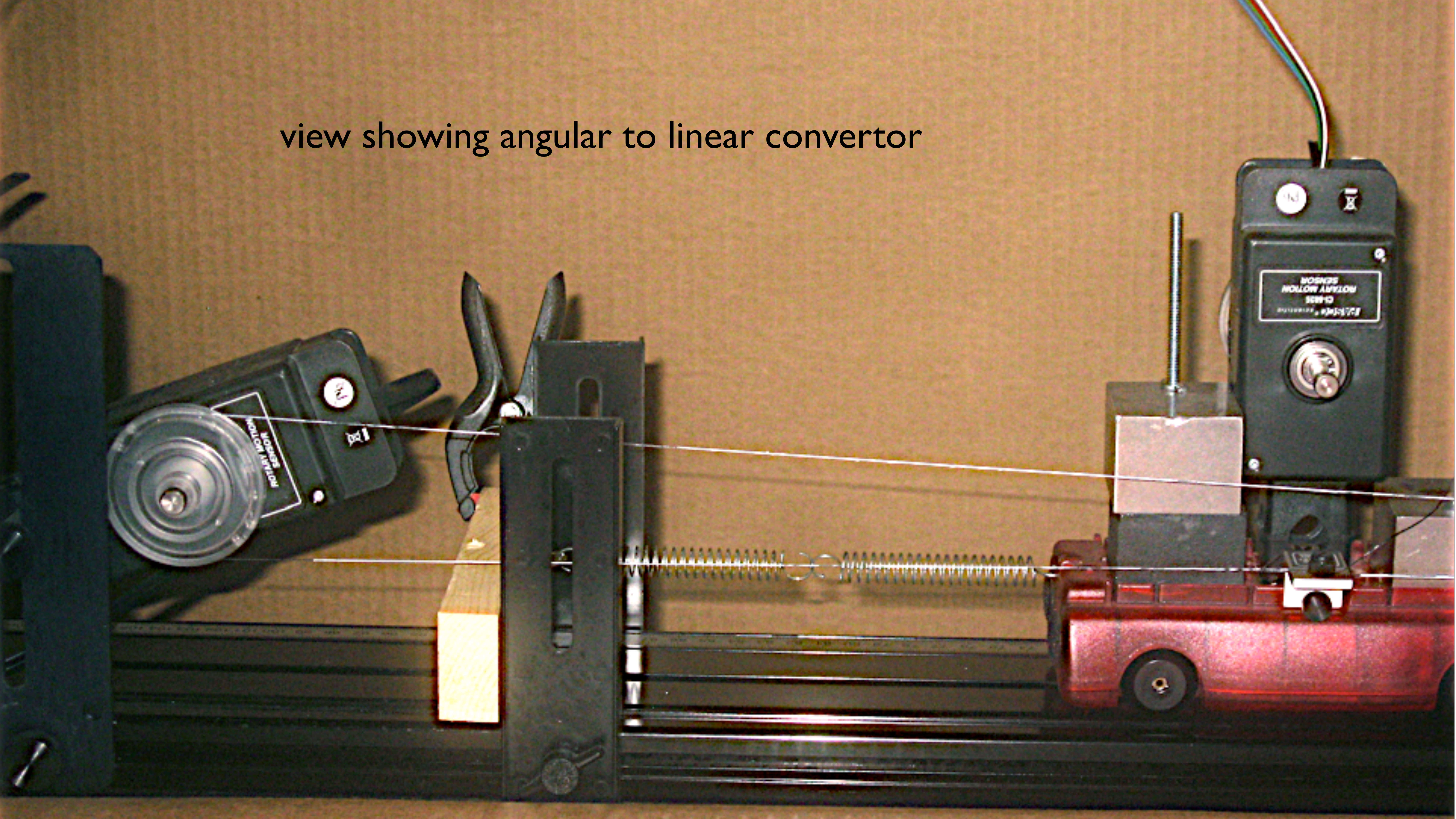




Six months later I purchased two rotary motion sensors, accessories, a track for the carts I'd won at an NCNAAPT raffle, and additional carts. That initial apparatus is pictured below.



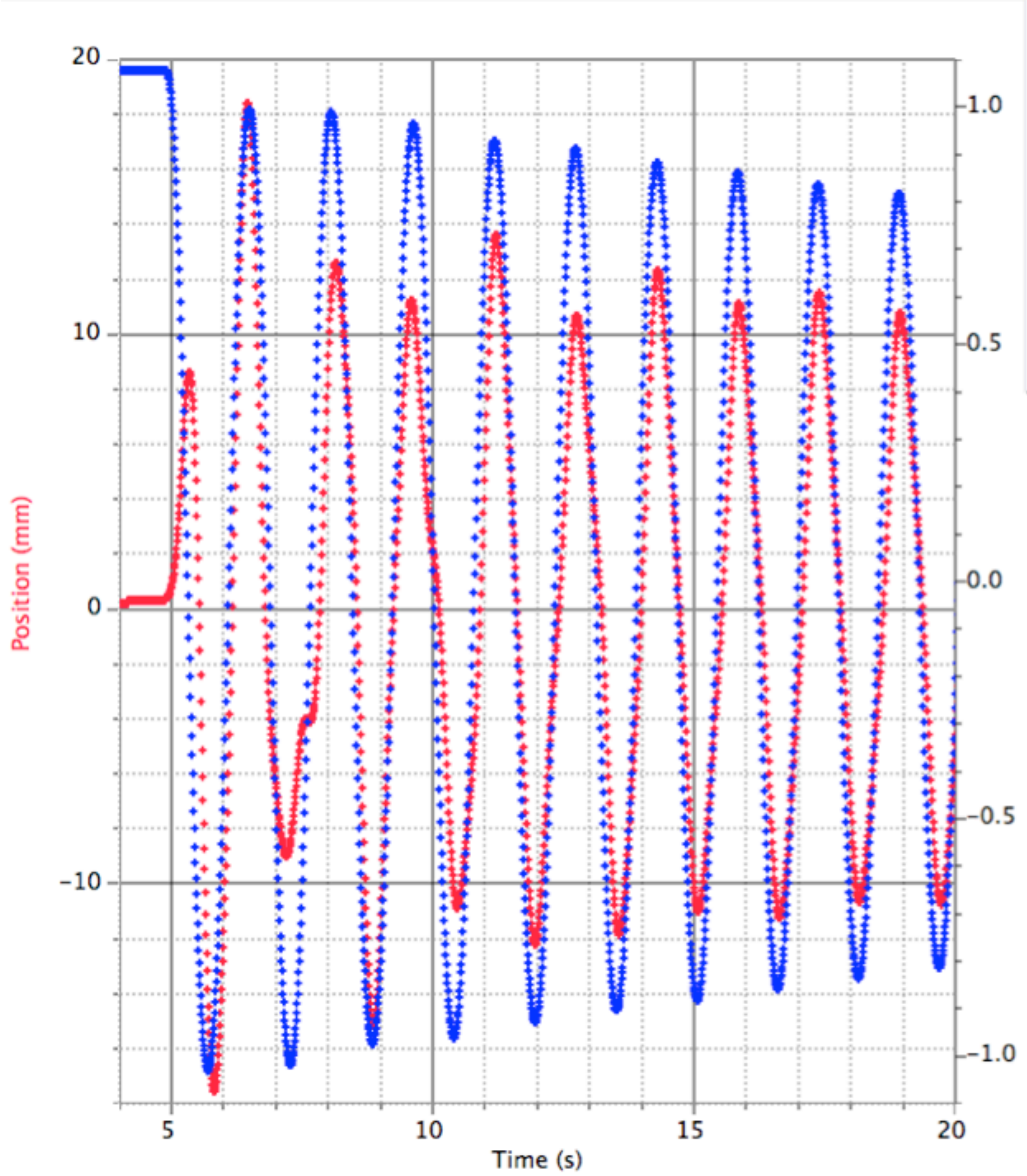
Note the partially obscured outrigger friction pad.

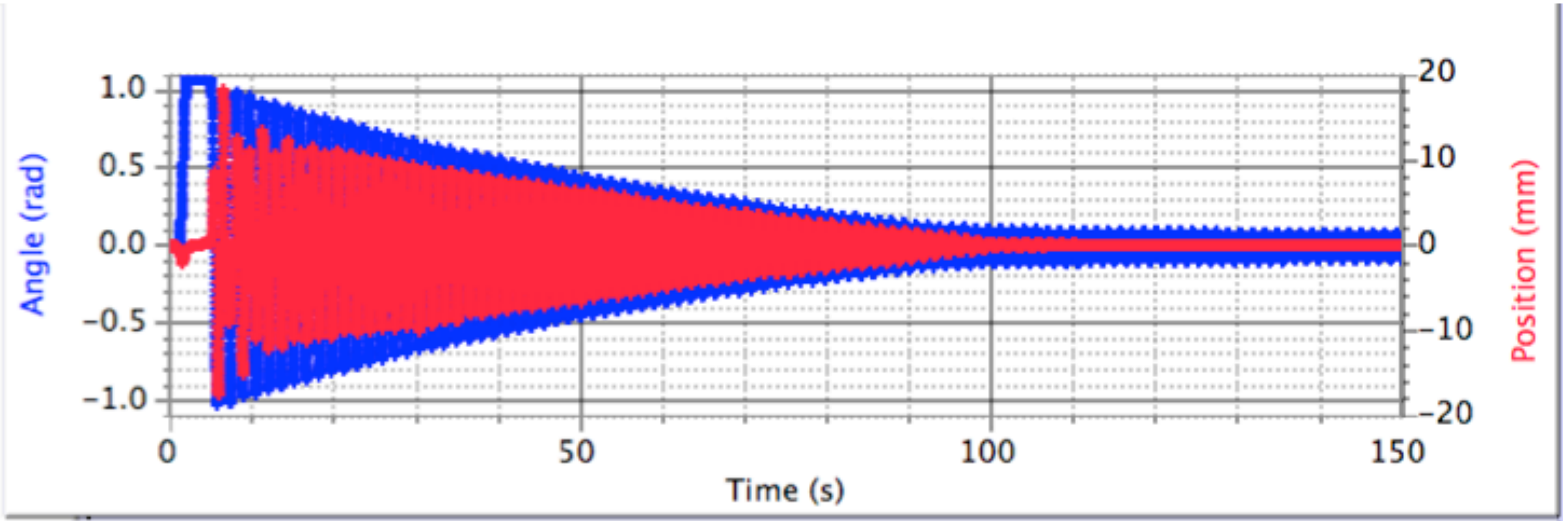


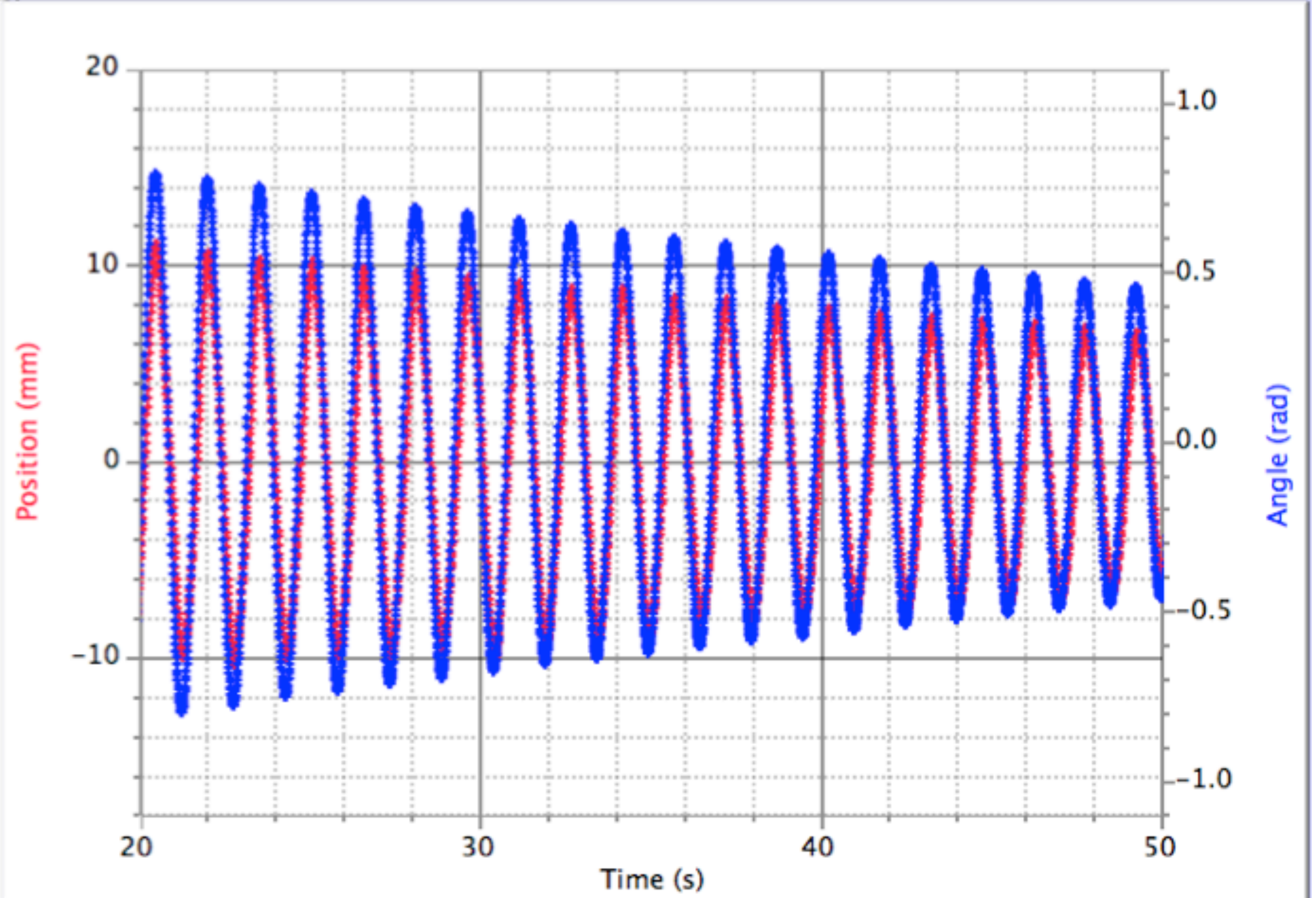
view showing angular to linear convertor

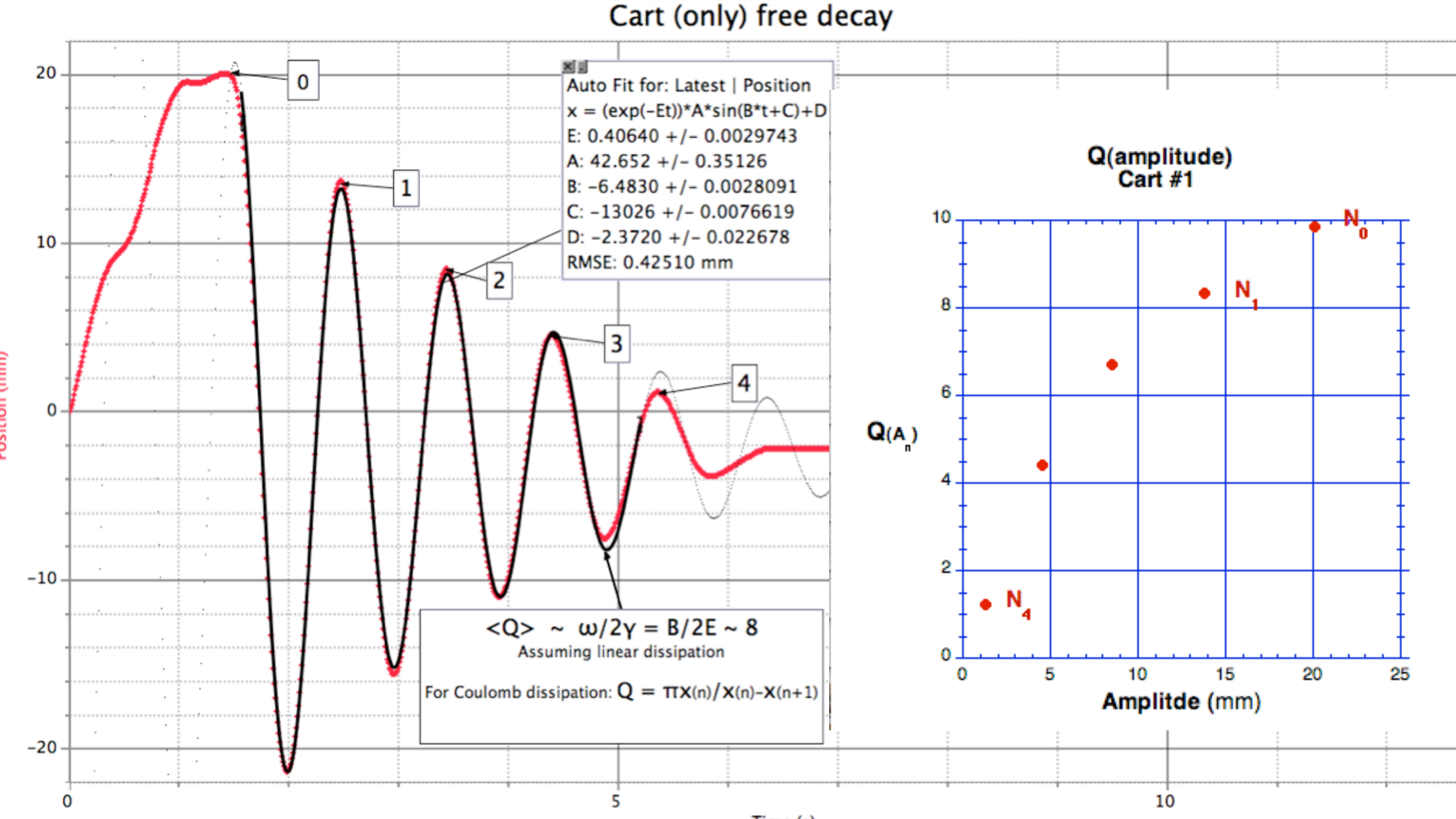
Speaking of the outrigger friction pad, the following are graphs of trials obtained with increasing pad friction. Included are measures of the damping found by displacing the cart with the pendulum removed.

Trial #1 (least damping)

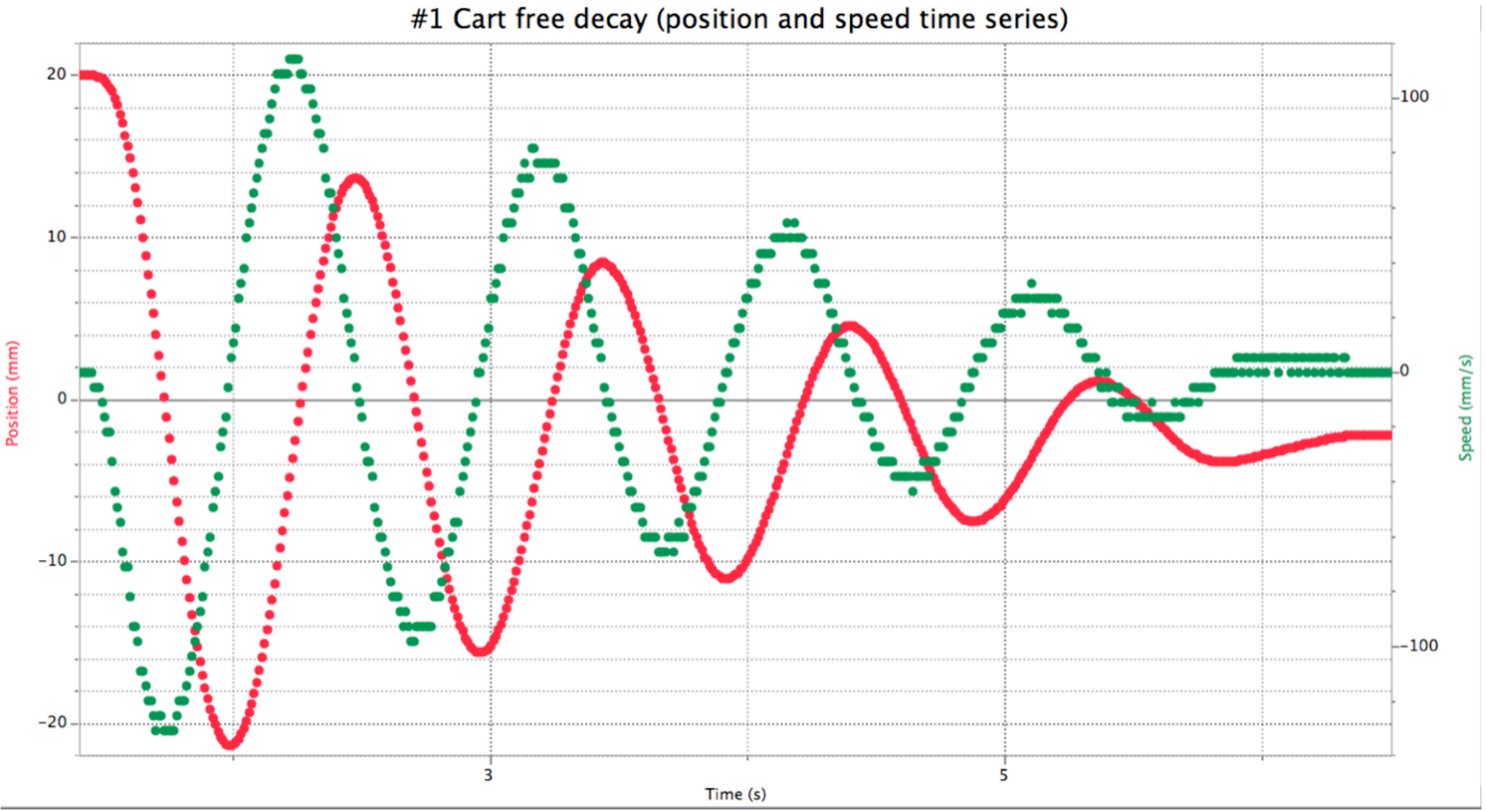


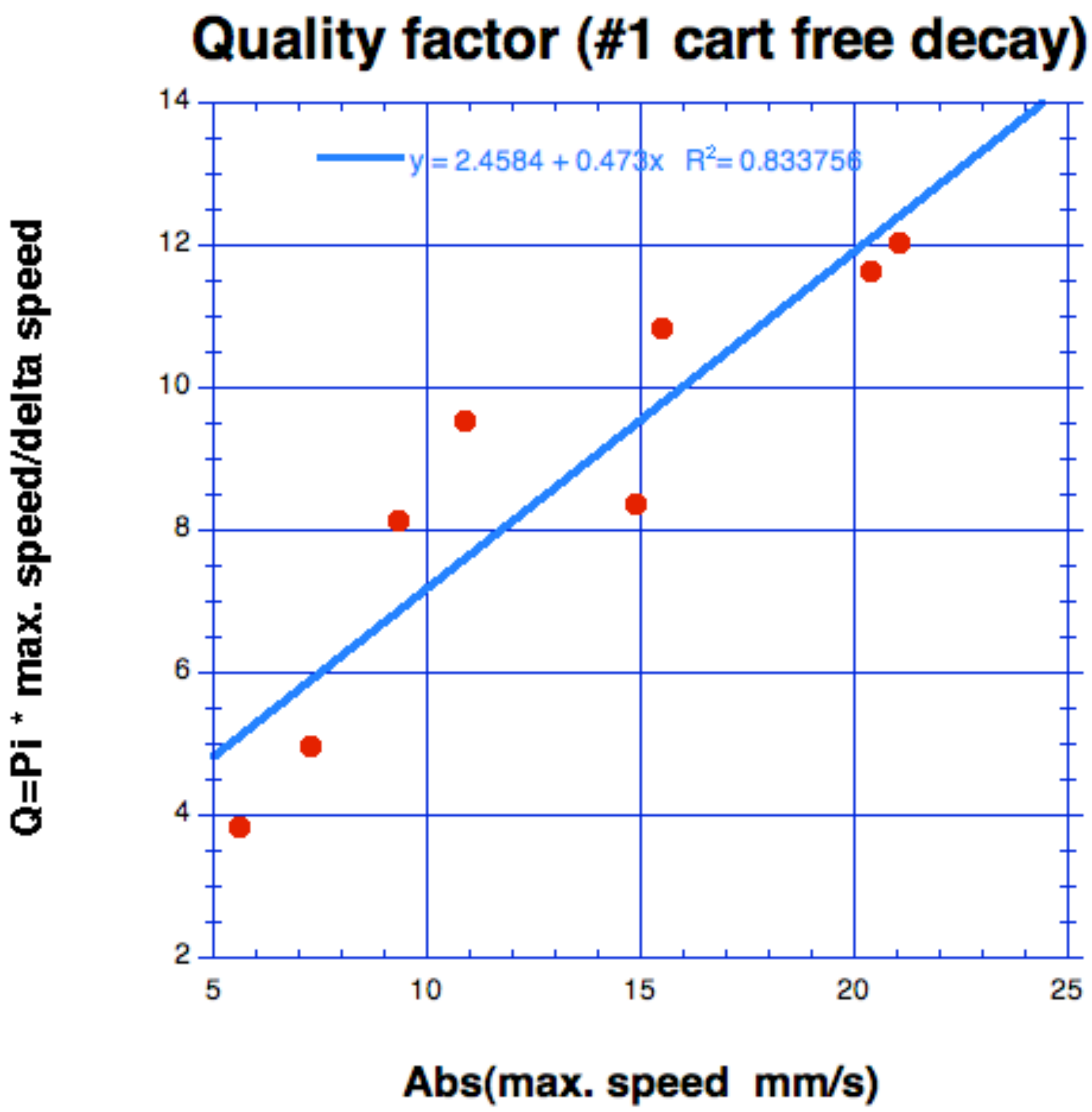





Cart (only) free decay
Auto Fit for: Latest | Position
x = (exp(-Et))*A*sin(B*t+C)+D
E: 0.40640 +/- 0.0029743
A: 42.652 +/- 0.35126
B: -6.4830 +/- 0.0028091
C: -13026 +/- 0.0076619
D: -2.3720 +/- 0.022678
RMSE: 0.42510 mm
0
1
2
3
4
<Q> ~ ω/2γ = B/2E ~ 8
Assuming linear dissipation
For Coulomb dissipation: Q = πx(n)/x(n)-x(n+1)
20
10
0
-10
-20
0
5
10
Q(amplitude)
Cart #1
$Q_{(A_n)}$
N_0
N_1
N_4
Amplitde (mm)
0
2
4
6
8
10
0
5
10
15
20
25

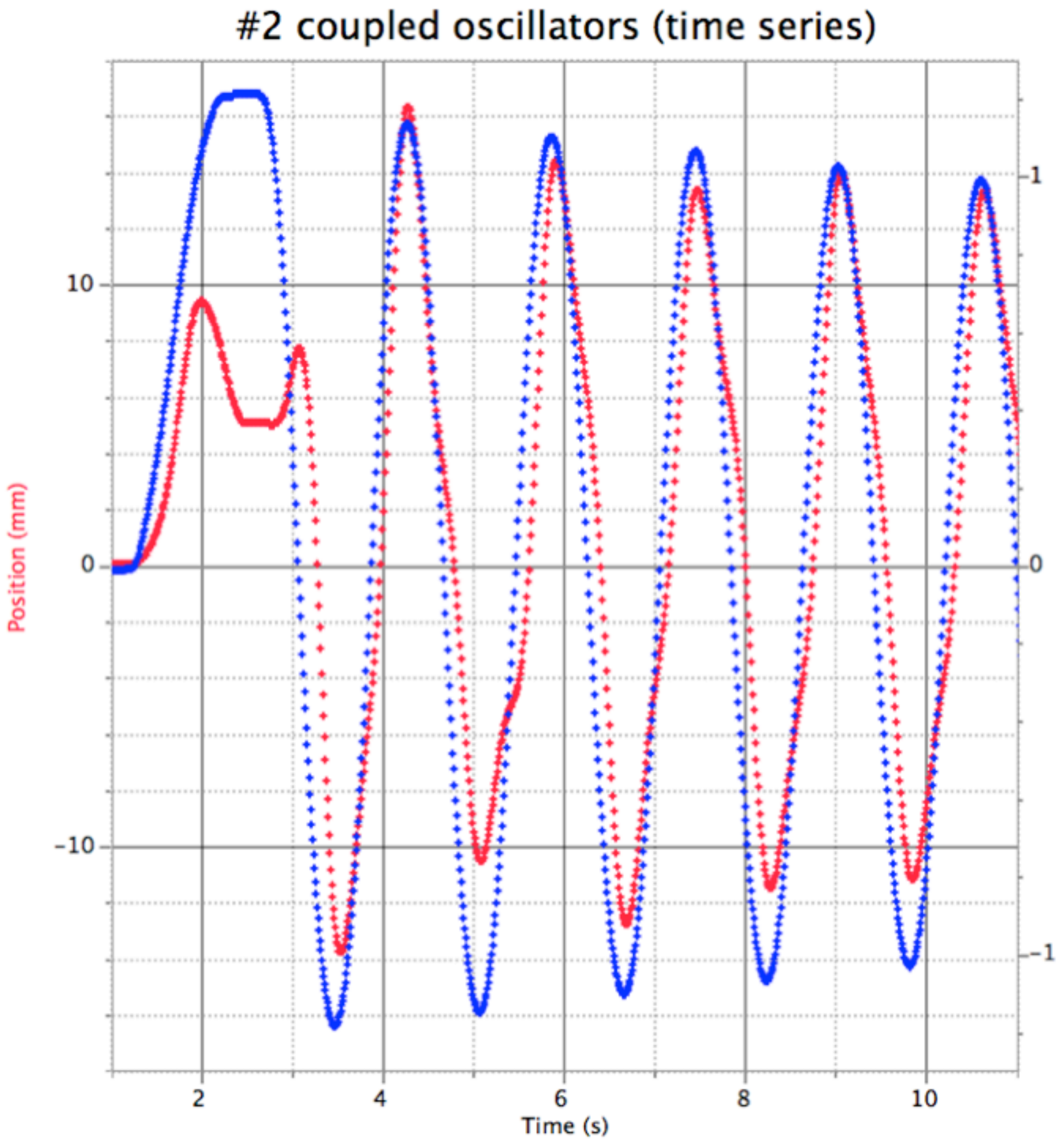
Note the problem above with the calculation of the Quality factor.
Corrected below:

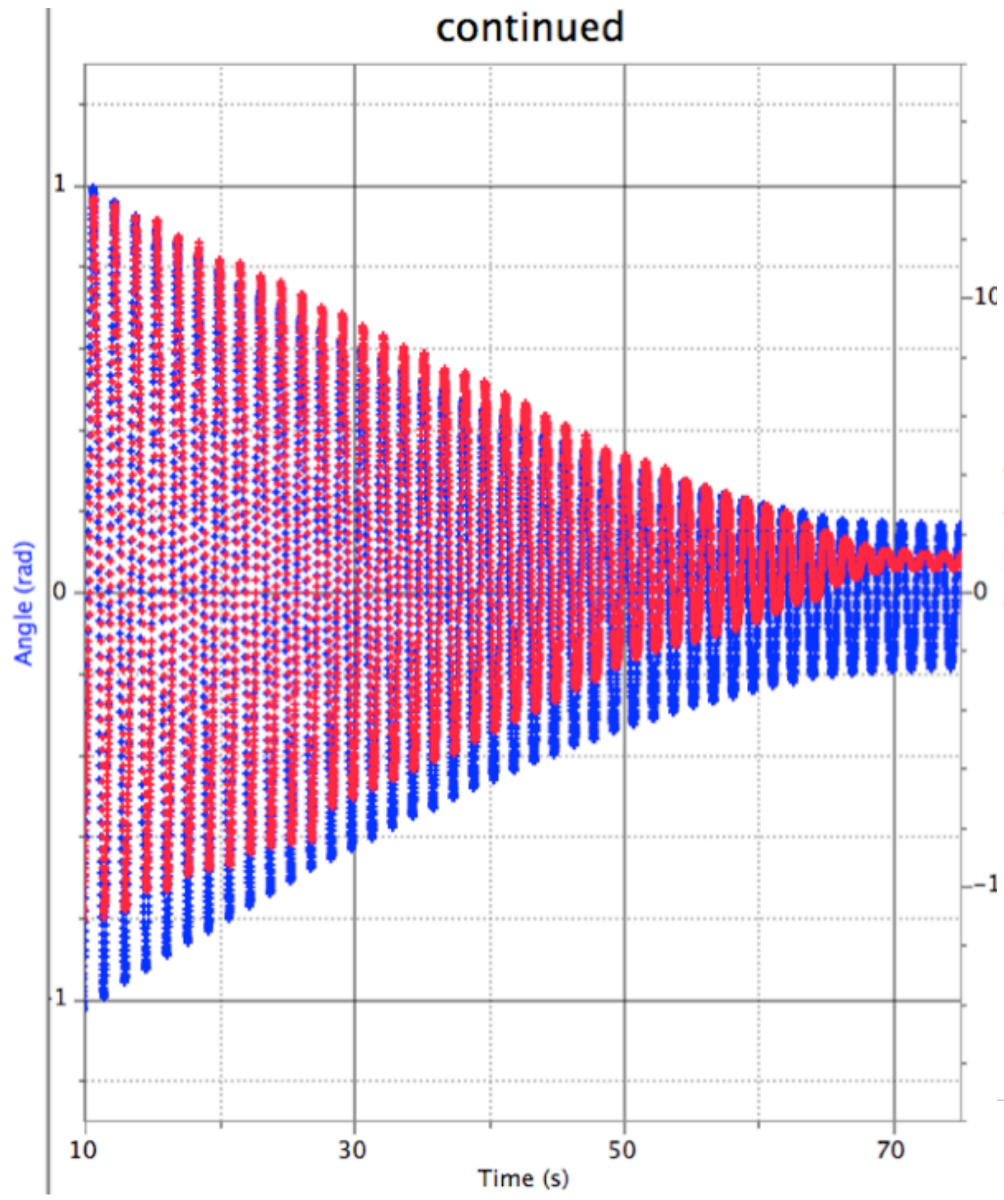


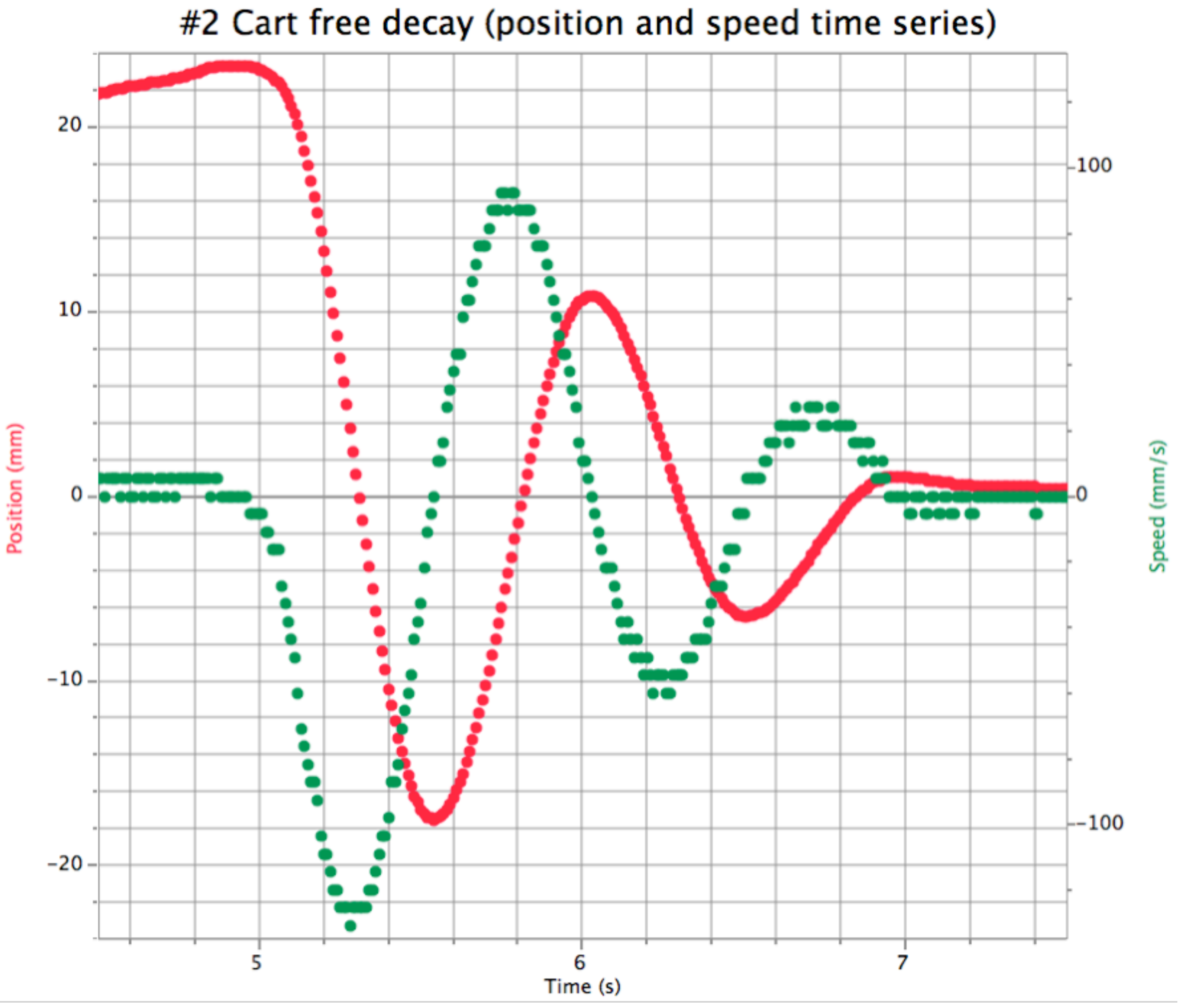


Trial #2

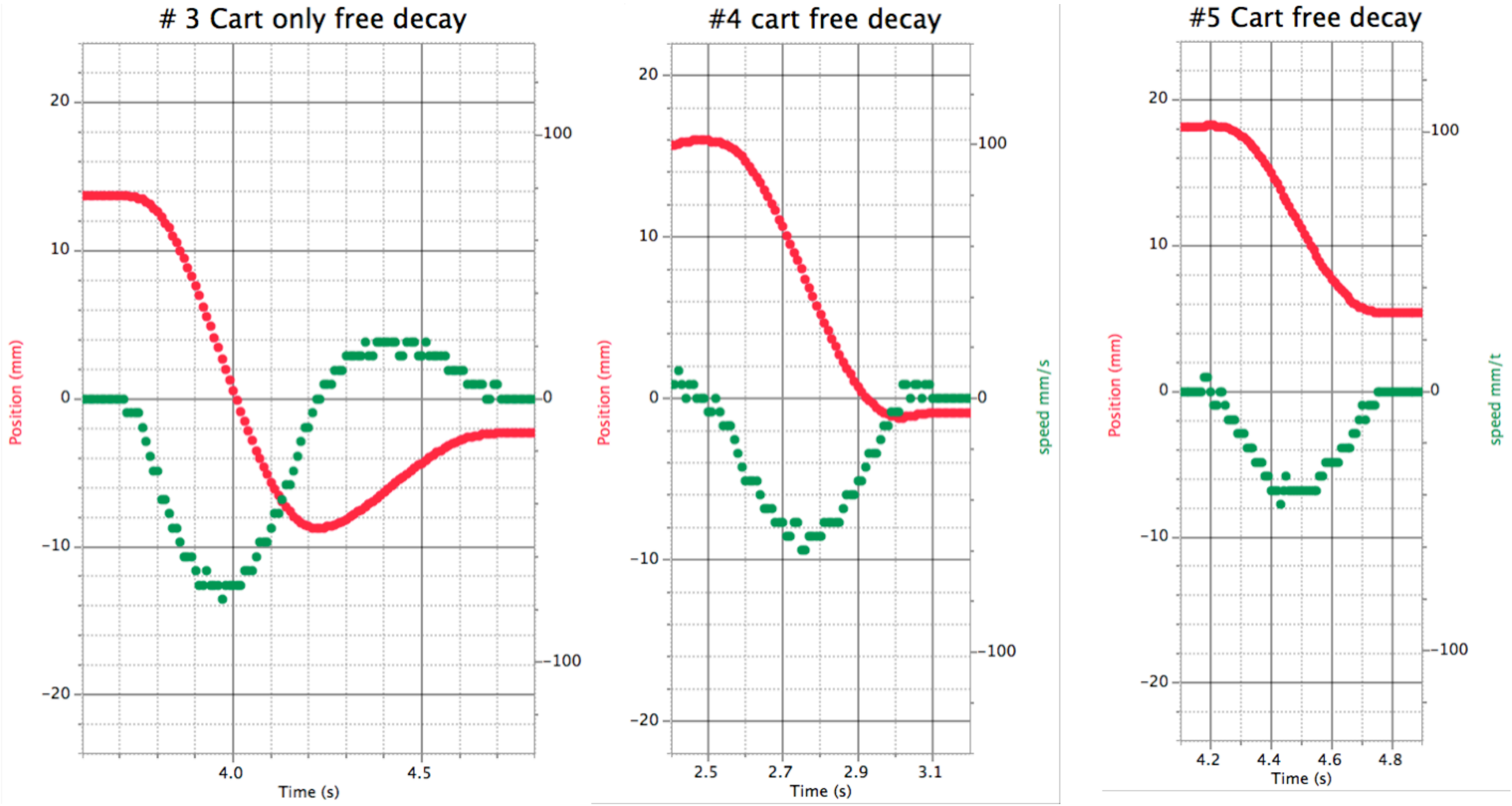
Unfortunately, I didn't hold the cart at its equilibrium position until I released the displaced the pendulum (for trial #2 only).

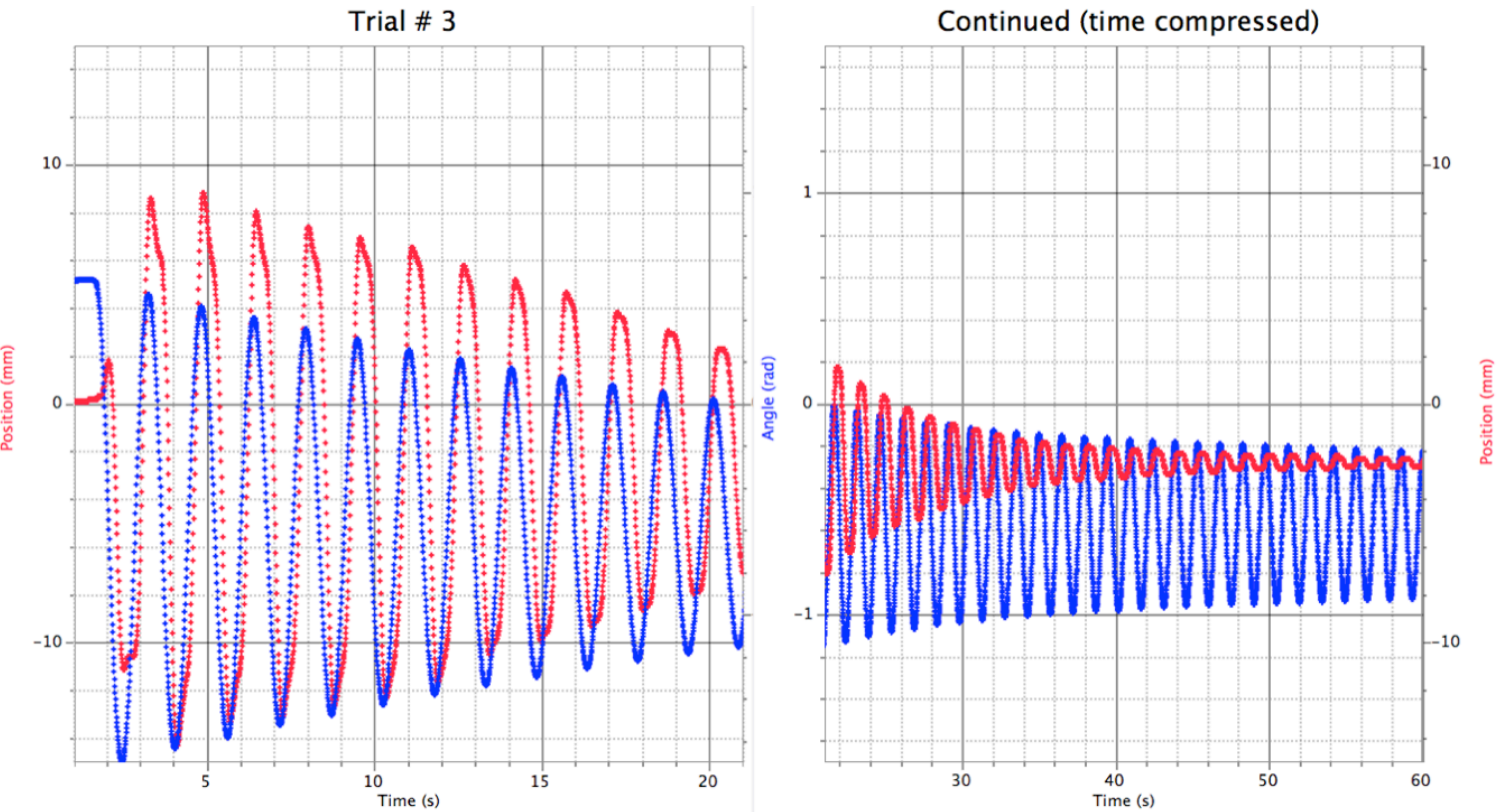




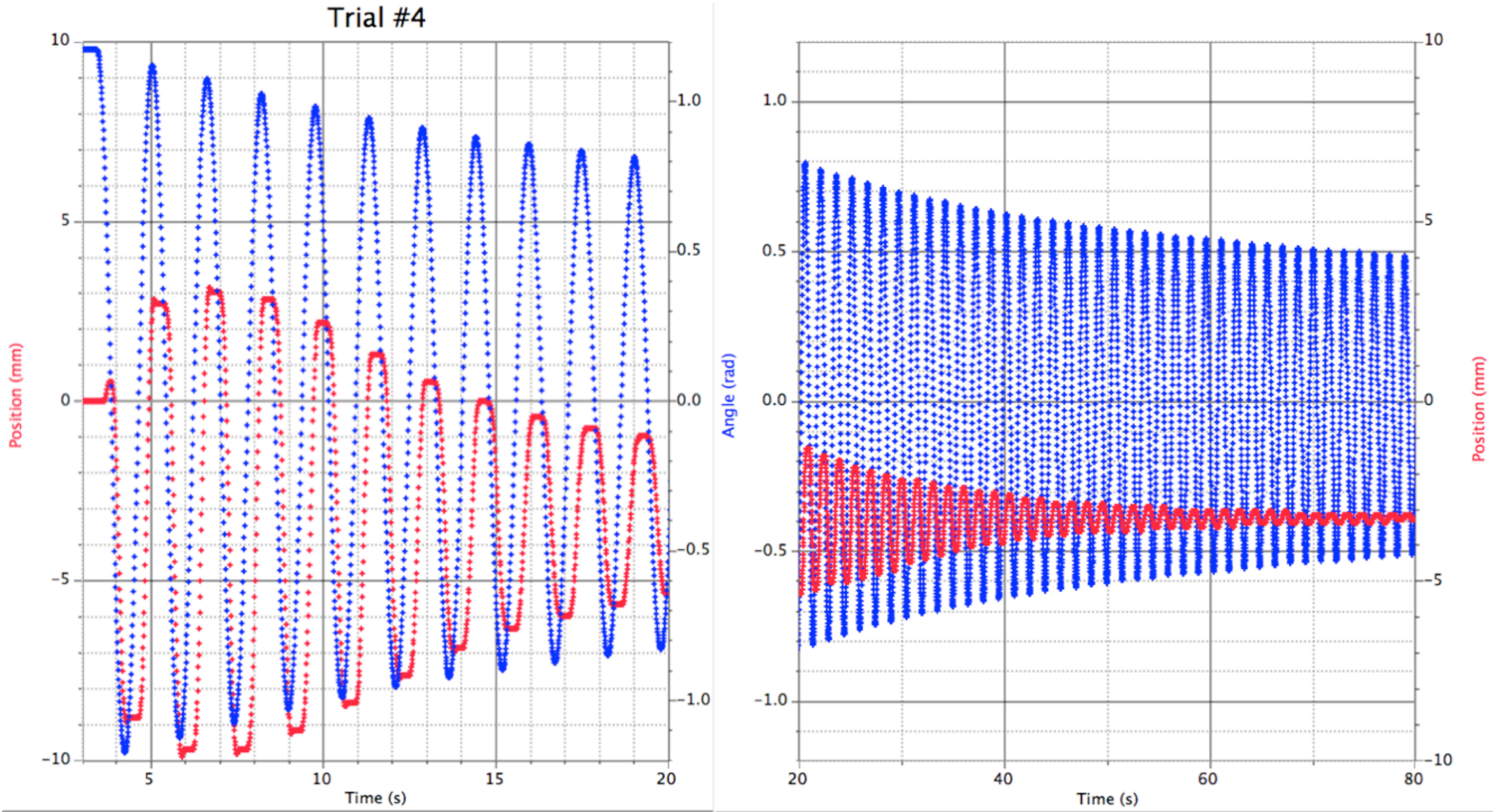


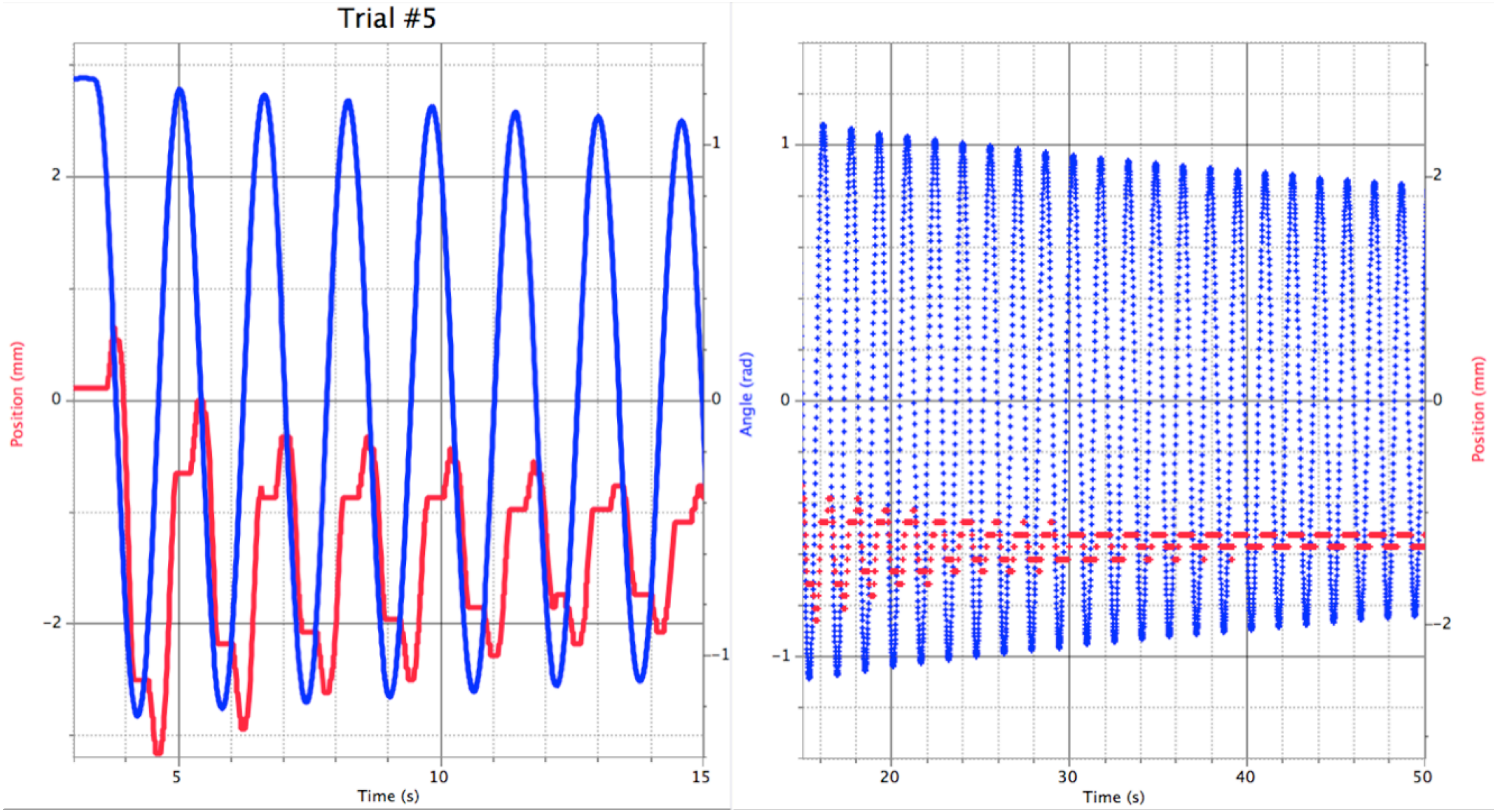
Approximately normalized's free decay graphs showing the effect of increasing Coulomb friction



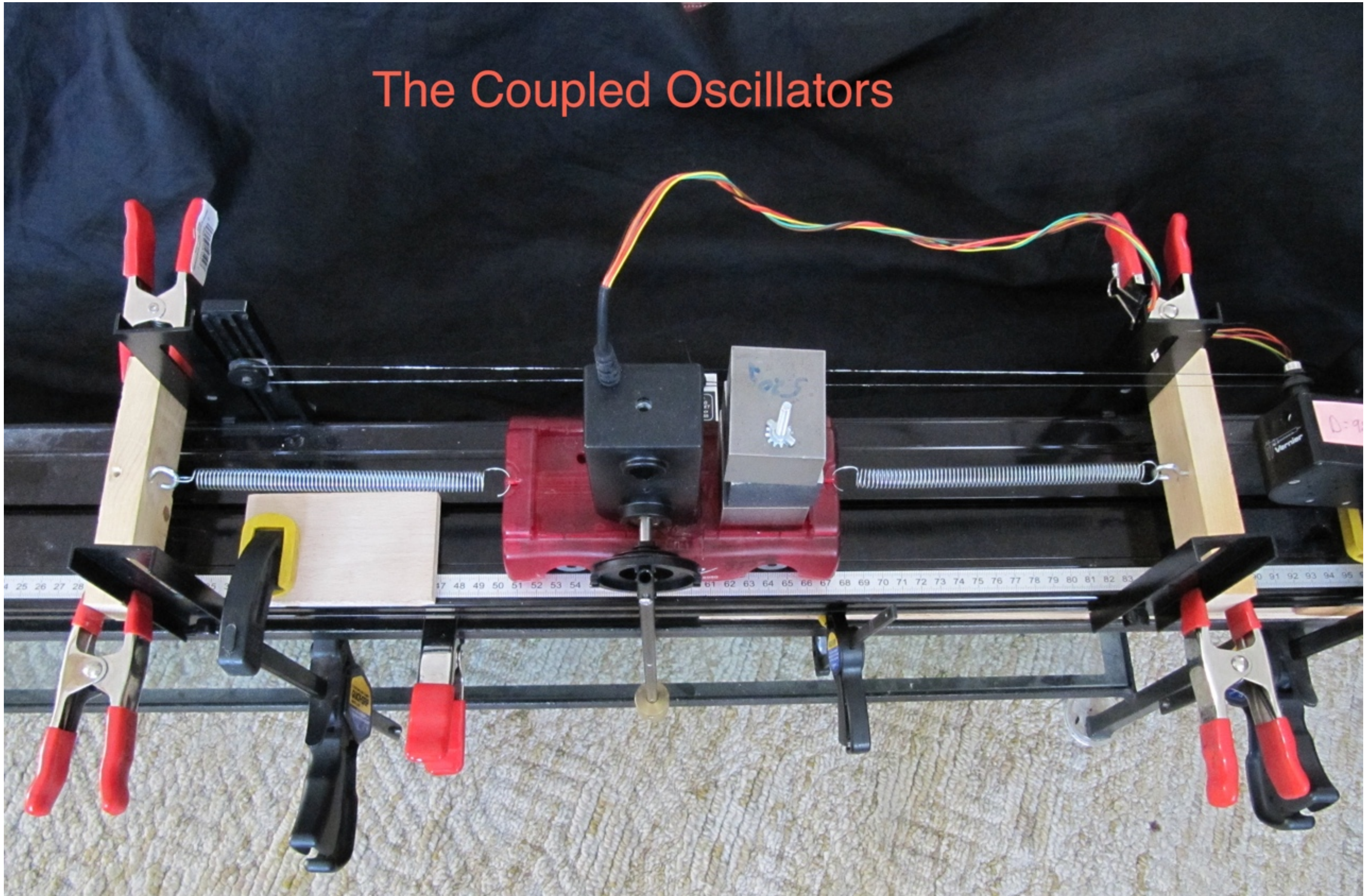


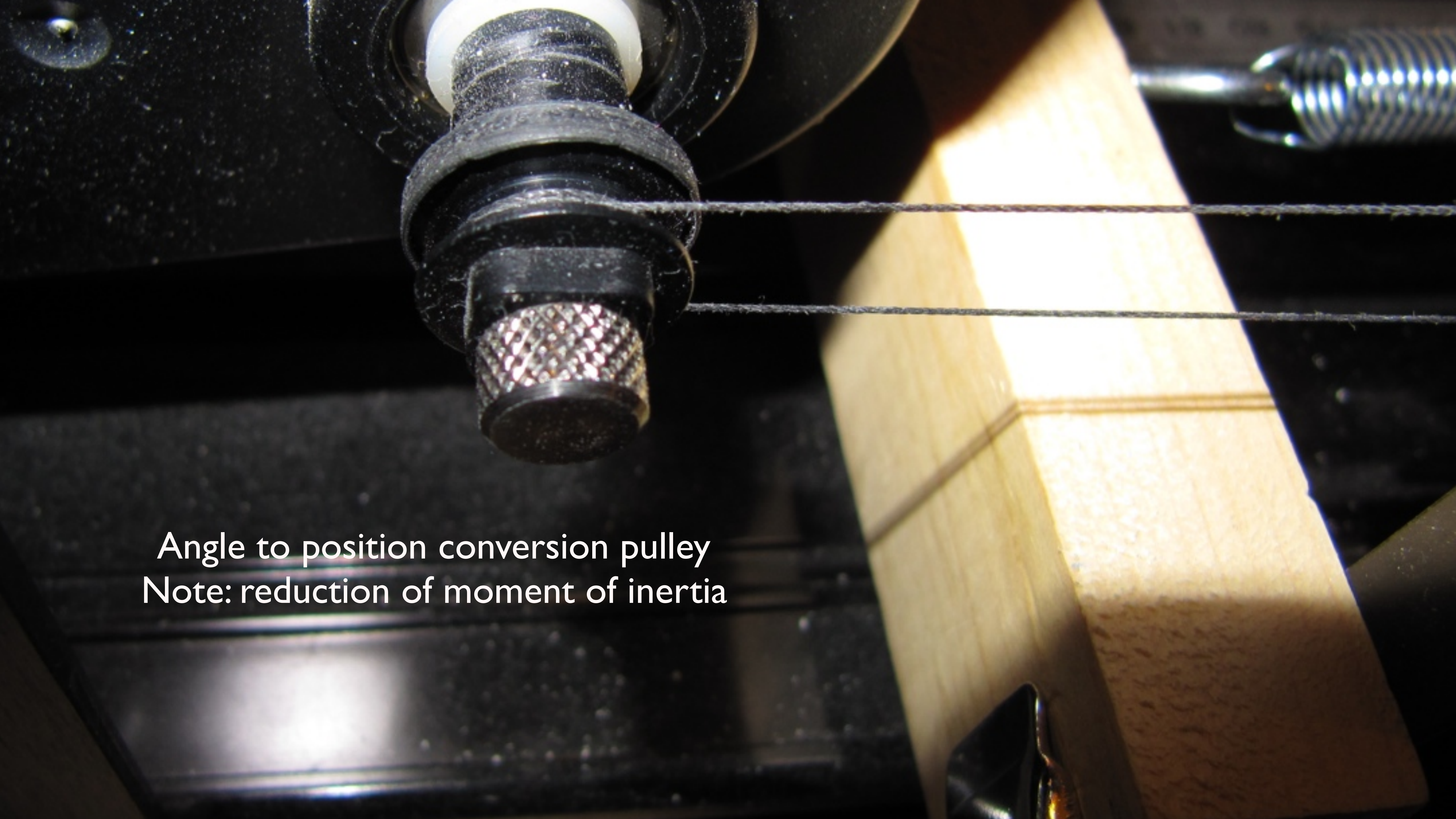
Trial # 3
Continued (time compressed)
Position (mm)
Angle (rad)
Position (mm)
10
0
-10
1
0
-1
10
0
-10
5
10
15
20
30
40
50
60
Time (s)
Time (s)


Trial #4
Position (mm)
Angle (rad)
Time (s)
Angle (rad)
Position (mm)
Time (s)

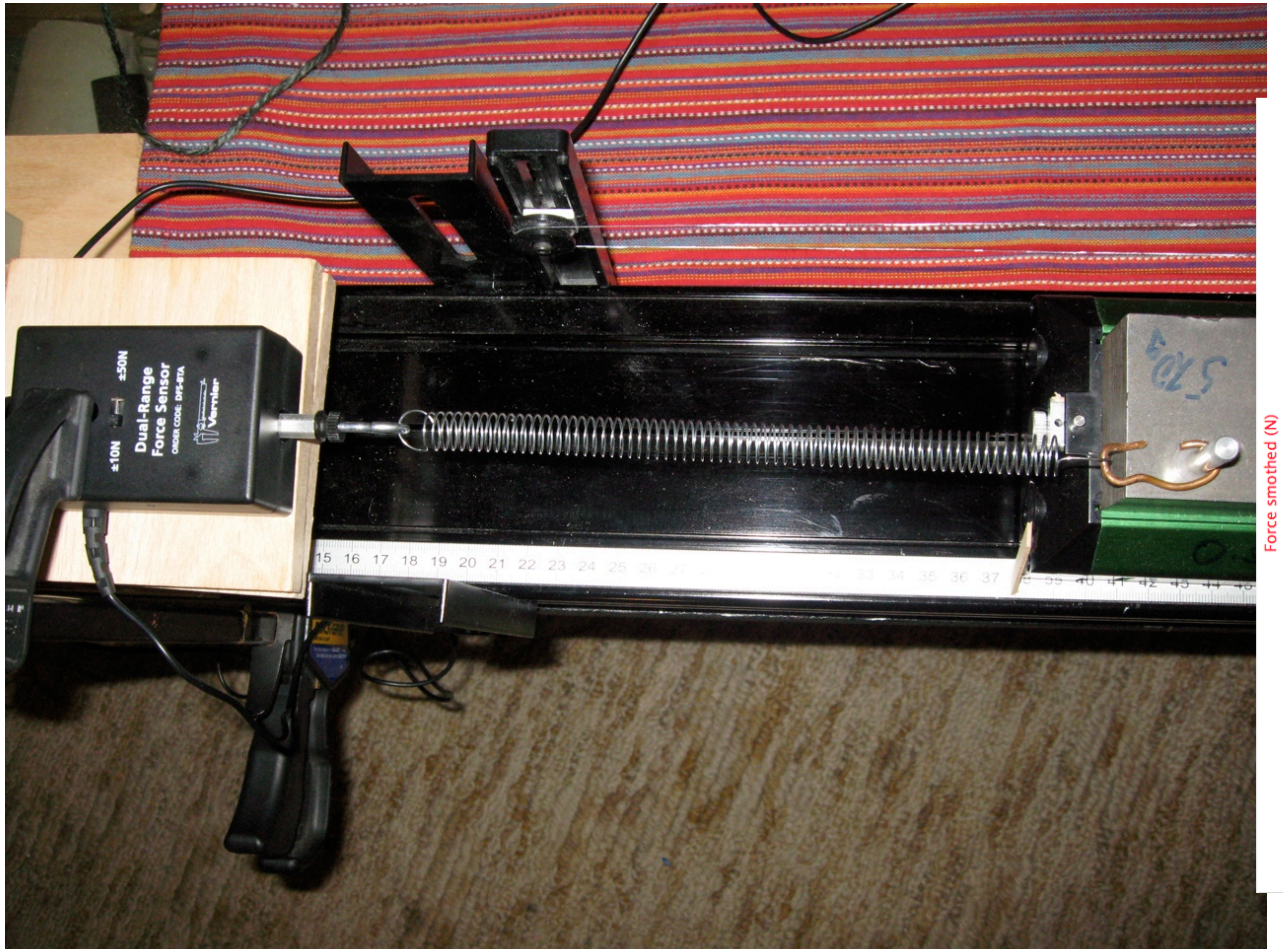

Trial #5
Position (mm)
Time (s)
Angle (rad)
Position (mm)
Time (s)

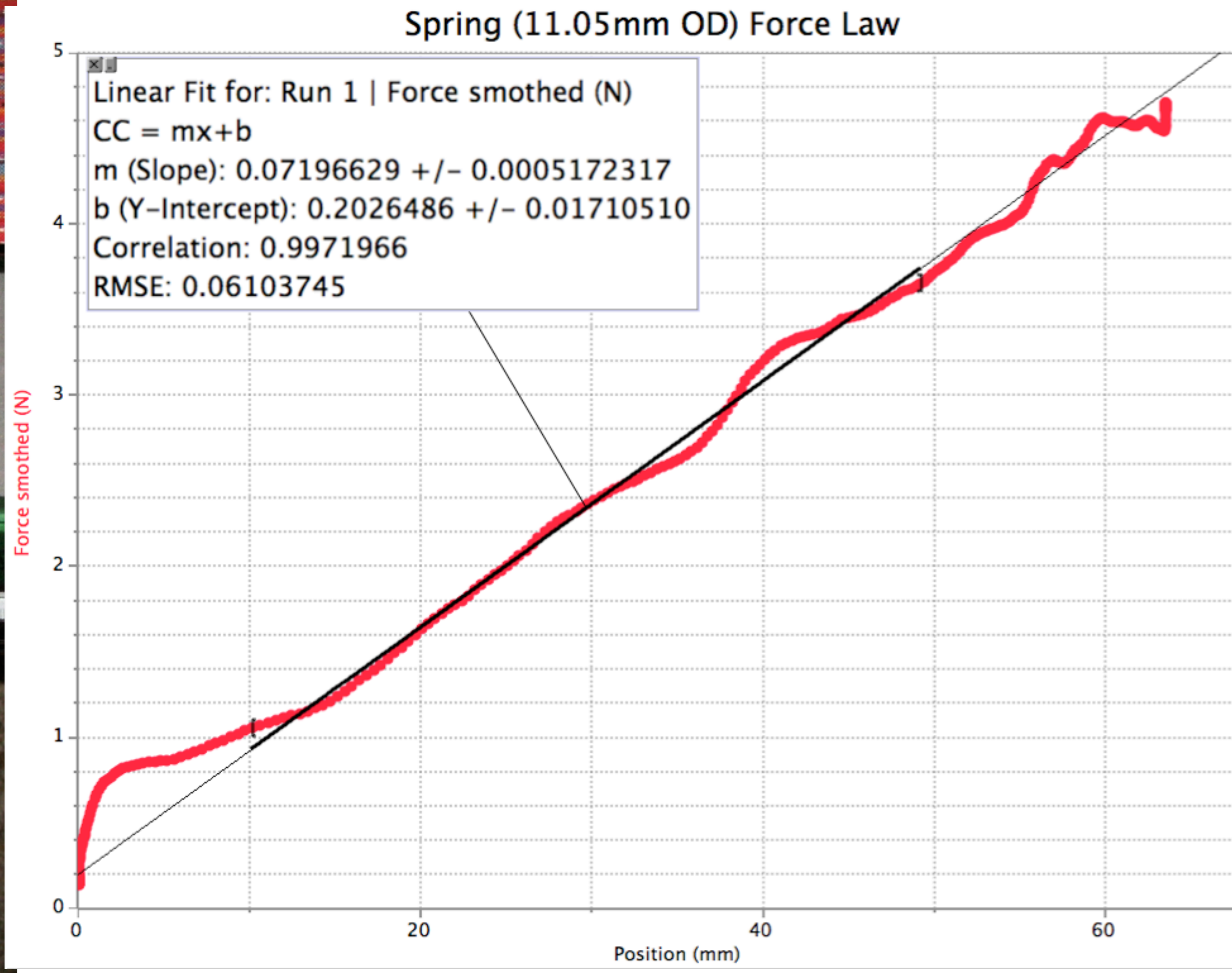
The New Apparatus




Angle to position conversion pulley
Note: reduction of moment of inertia

Finding the spring's force constant.



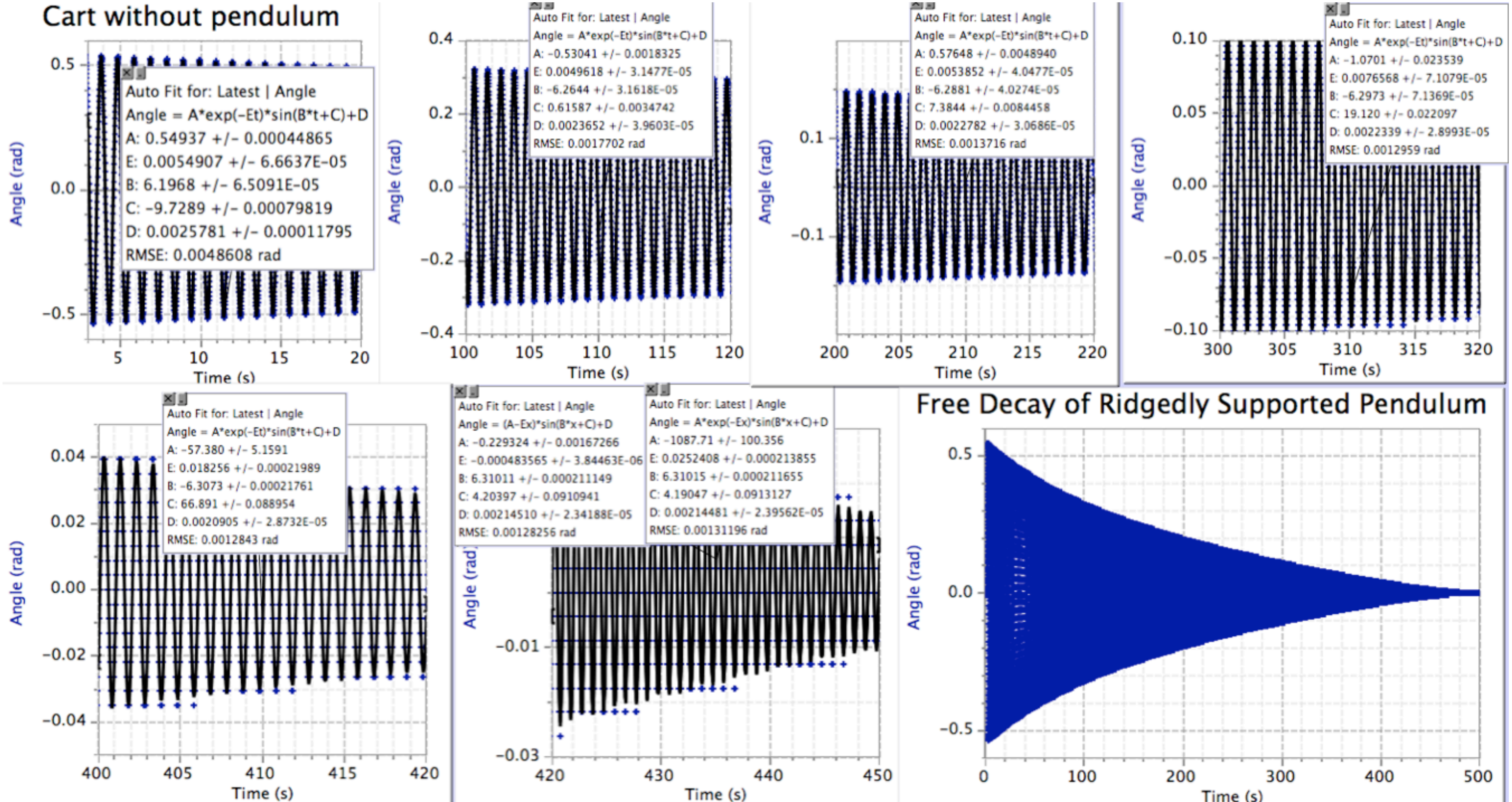


Photo' of previous apparatus and a graph of the present apparatus' force constant

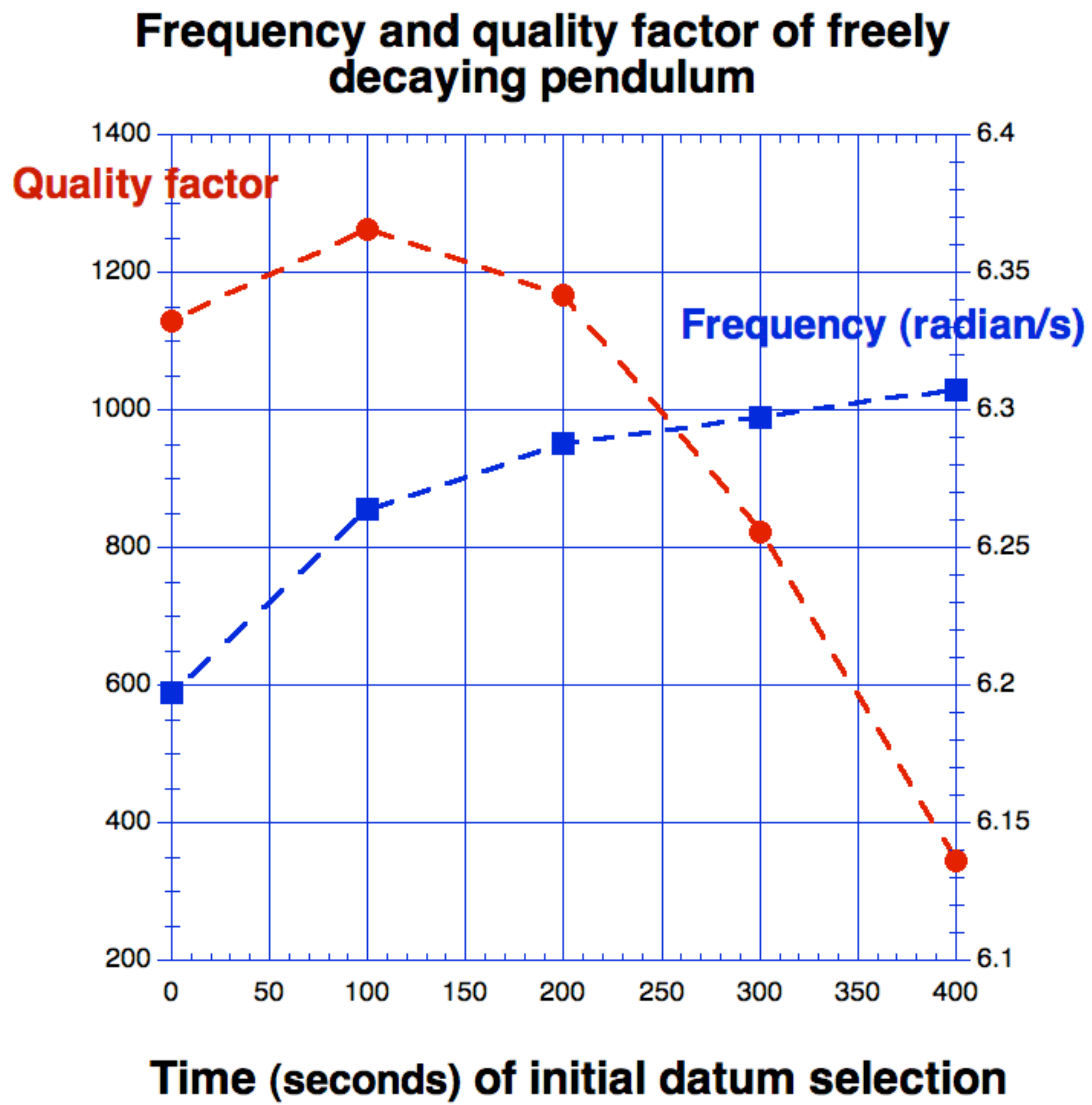
Note the apparent static friction.

The next five slides show the free decay of the pendulum with its support clamped to the track.

Finding the Pendulum's Frequency and Q from Its Free Decay



The multiple fits show the variation of both the frequency and the Q of the freely decaying pendulum. These are graphed below.



The variation of the Q is due to the changing dominance of the dissipation type. First quadratic, then linear, and finally Coulomb.

The Frequency variation is, of course, due to a pendulum's "circular error".

A superior method of showing the amount of each decay type is to directly measure the energy decay for each "beat". See: Siegel, et alii, Period-speed analysis of a pendulum

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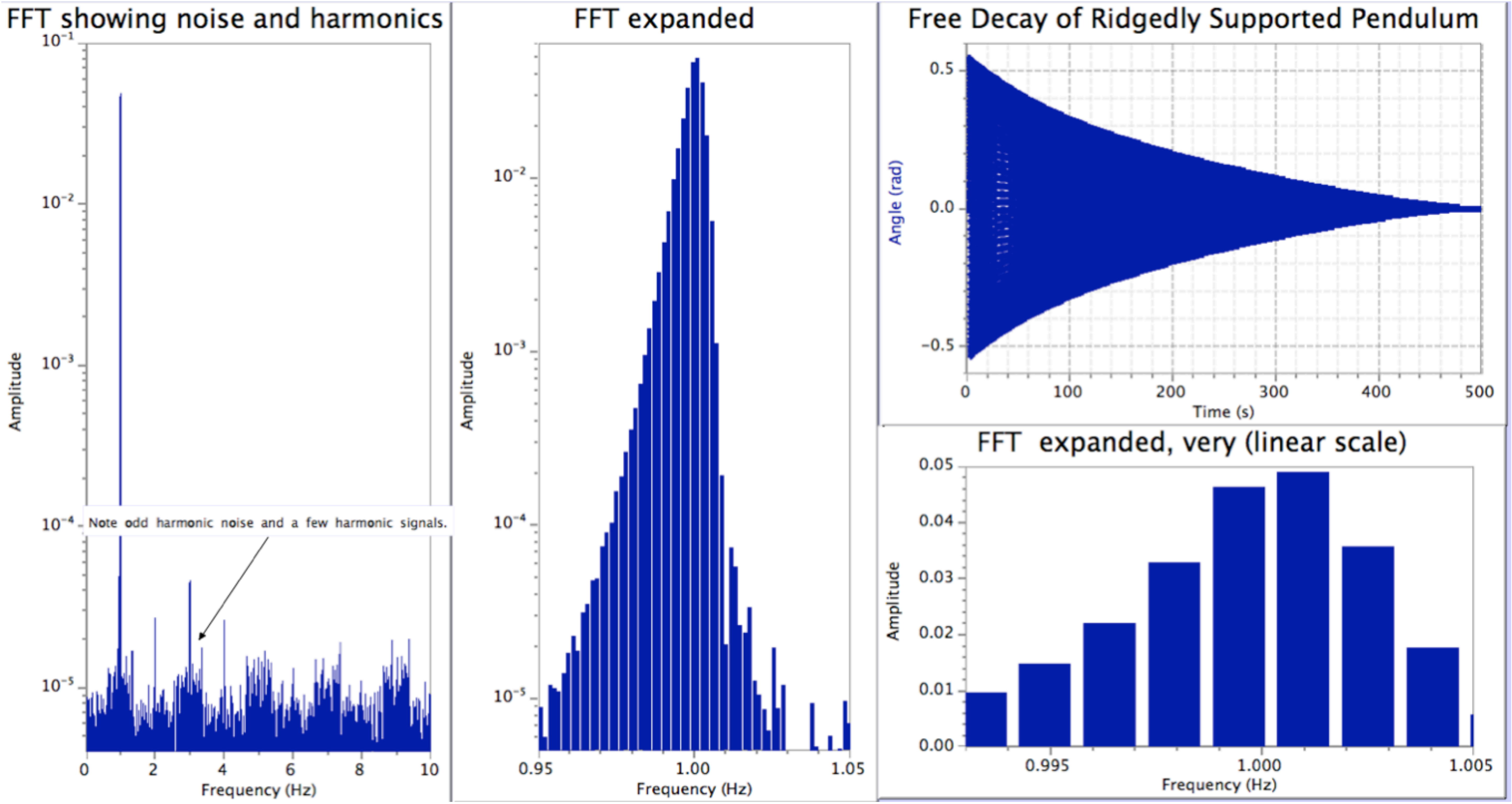
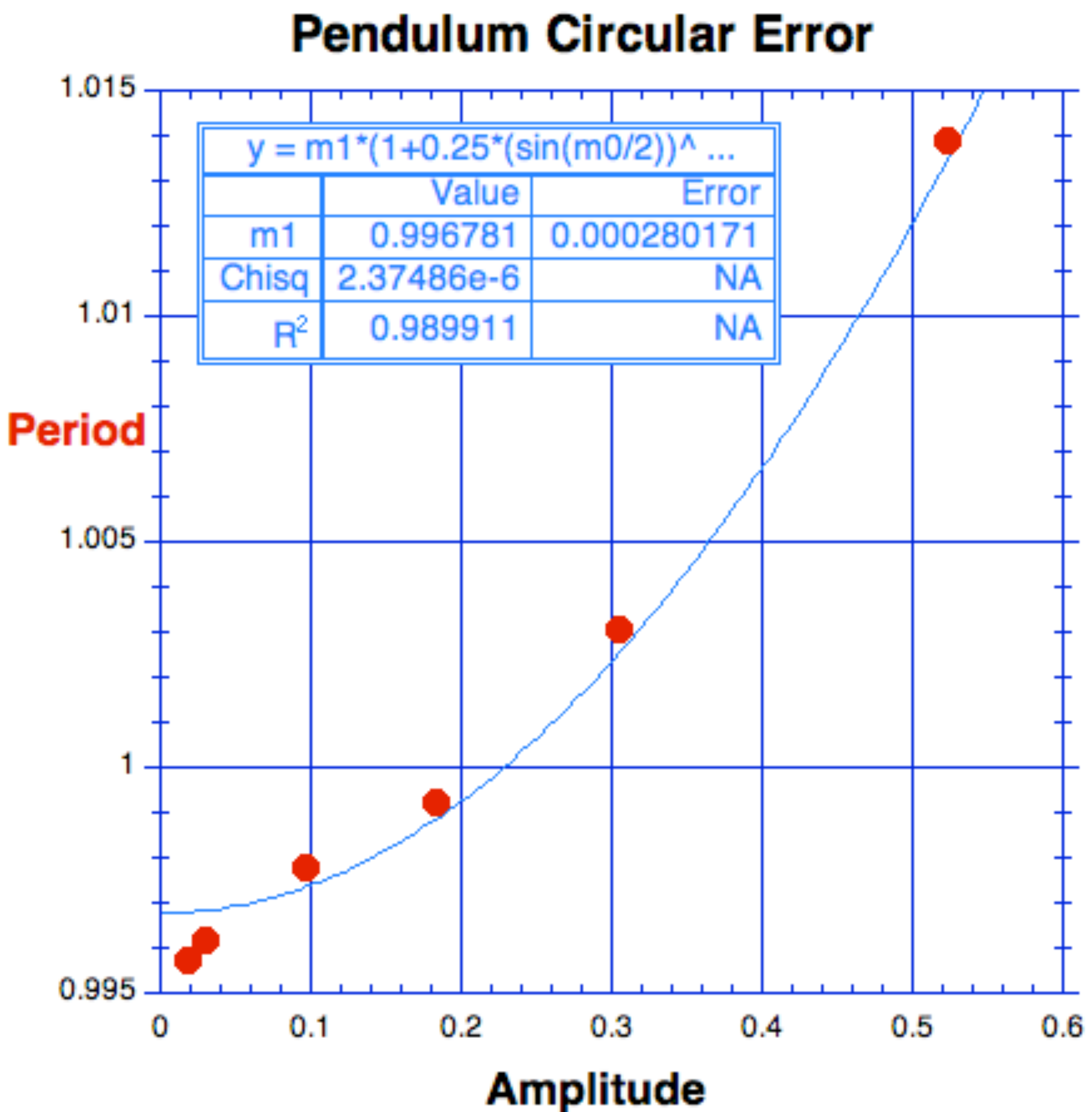
There is apparently "good" agreement between the previous graph of frequencies and the FFT ones below.

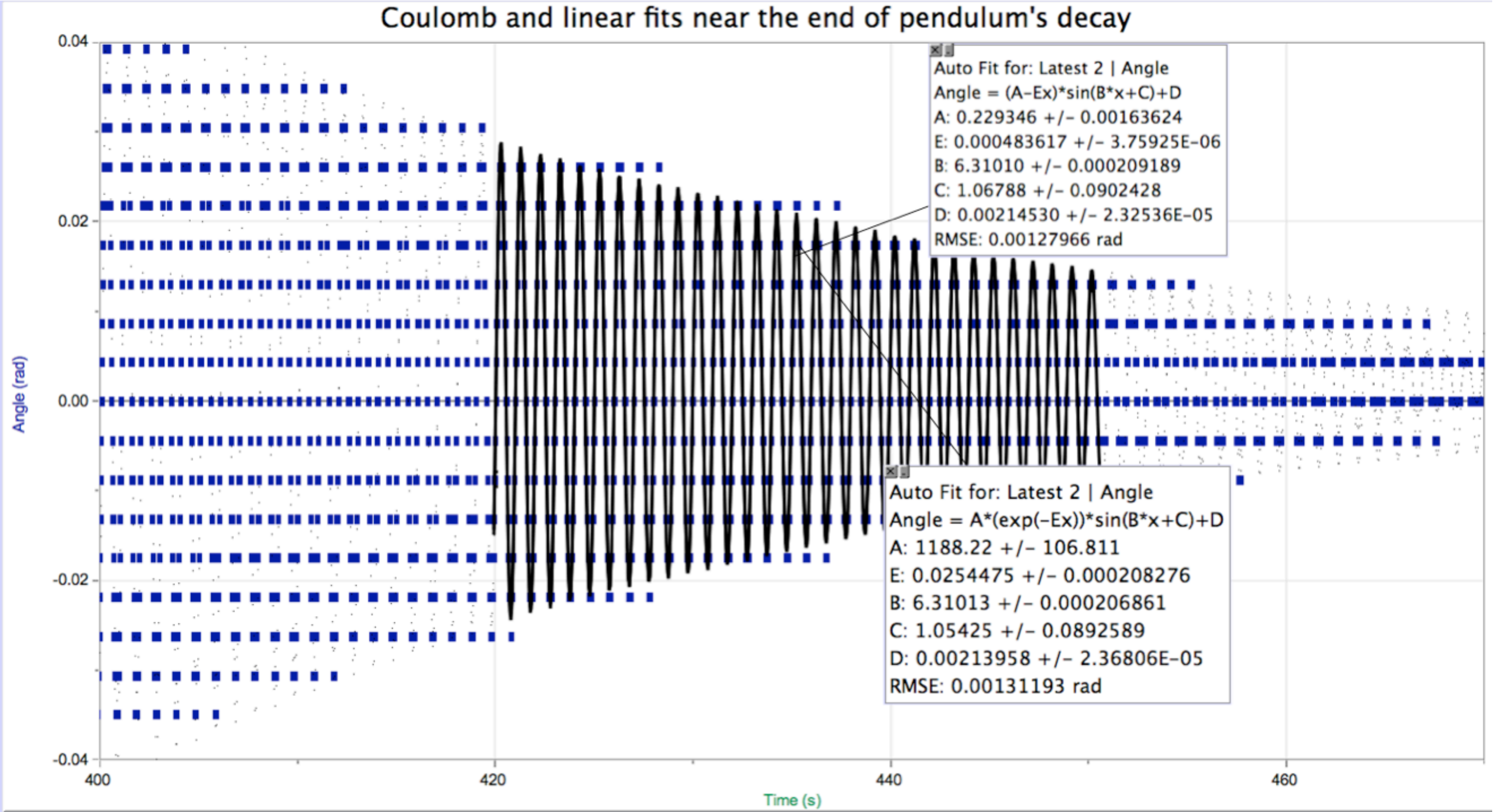


Here's a graph showing the circular error fitted to a first order correction. The discrepancy at low amplitude is likely due to the low resolution of the rotary motion sensor.

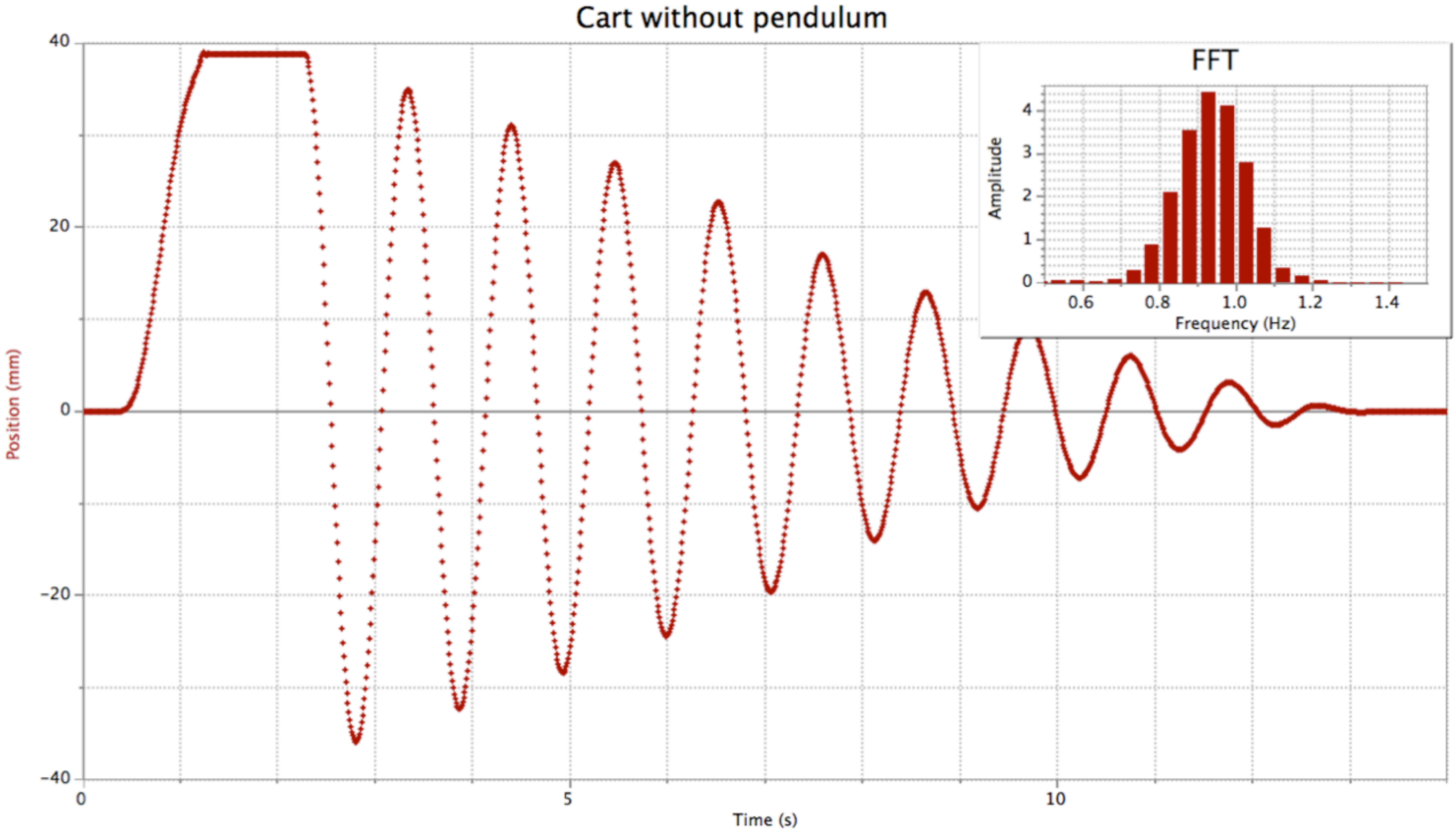
(The amplitude is in radians and the period in seconds.)



This graph illustrates the poor resolution and that the decay is somewhat constant.



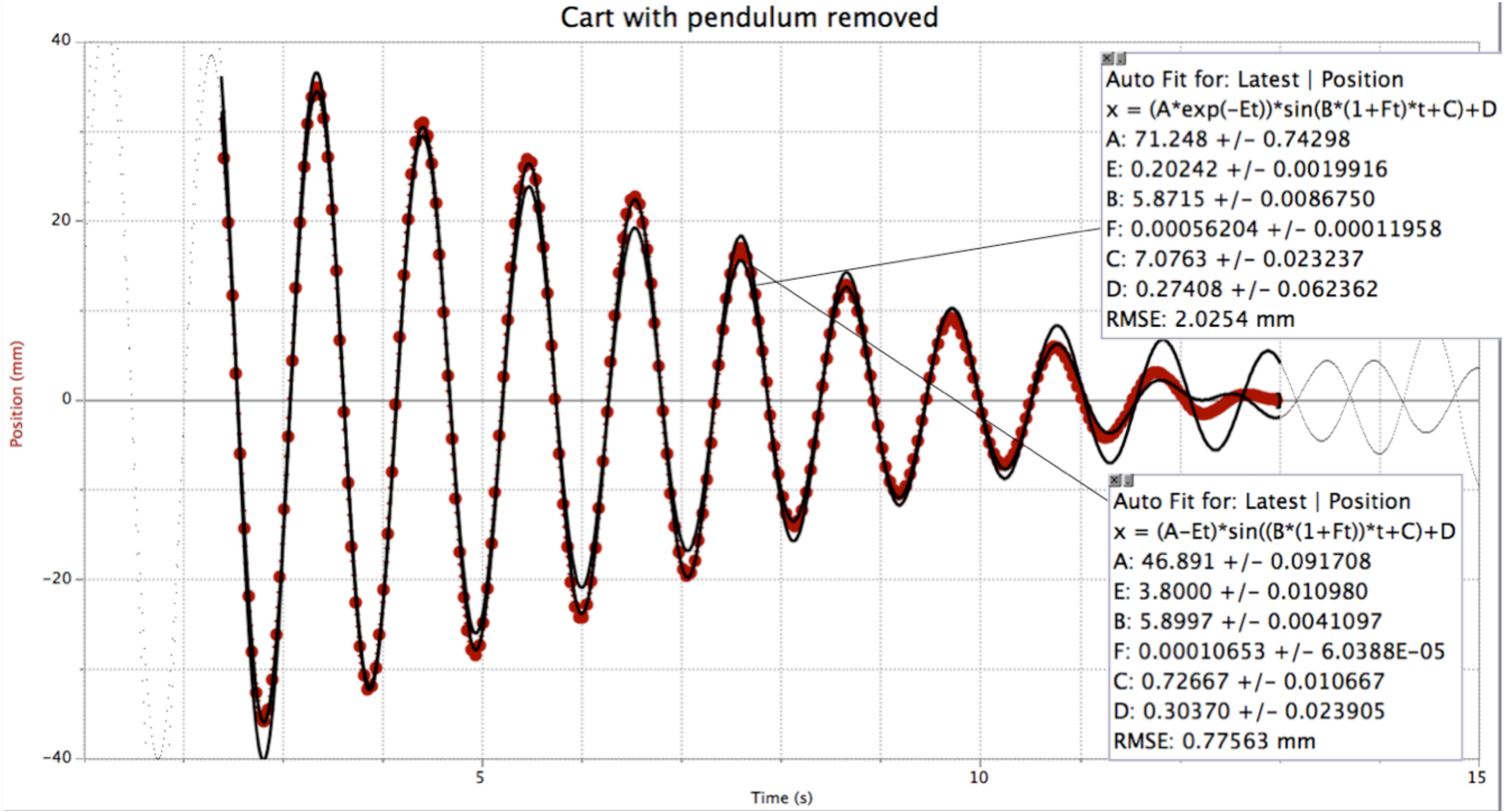
The following two slides show the cart's free decay with its pendulum removed.



Cart without pendulum
Position (mm)
Time (s)
40
20
0
-20
-40
0
5
10
FFT
Amplitude
Frequency (Hz)
4
3
2
1
0
0.6
0.8
1.0
1.2
1.4

Q (from the linear fit) = 29.

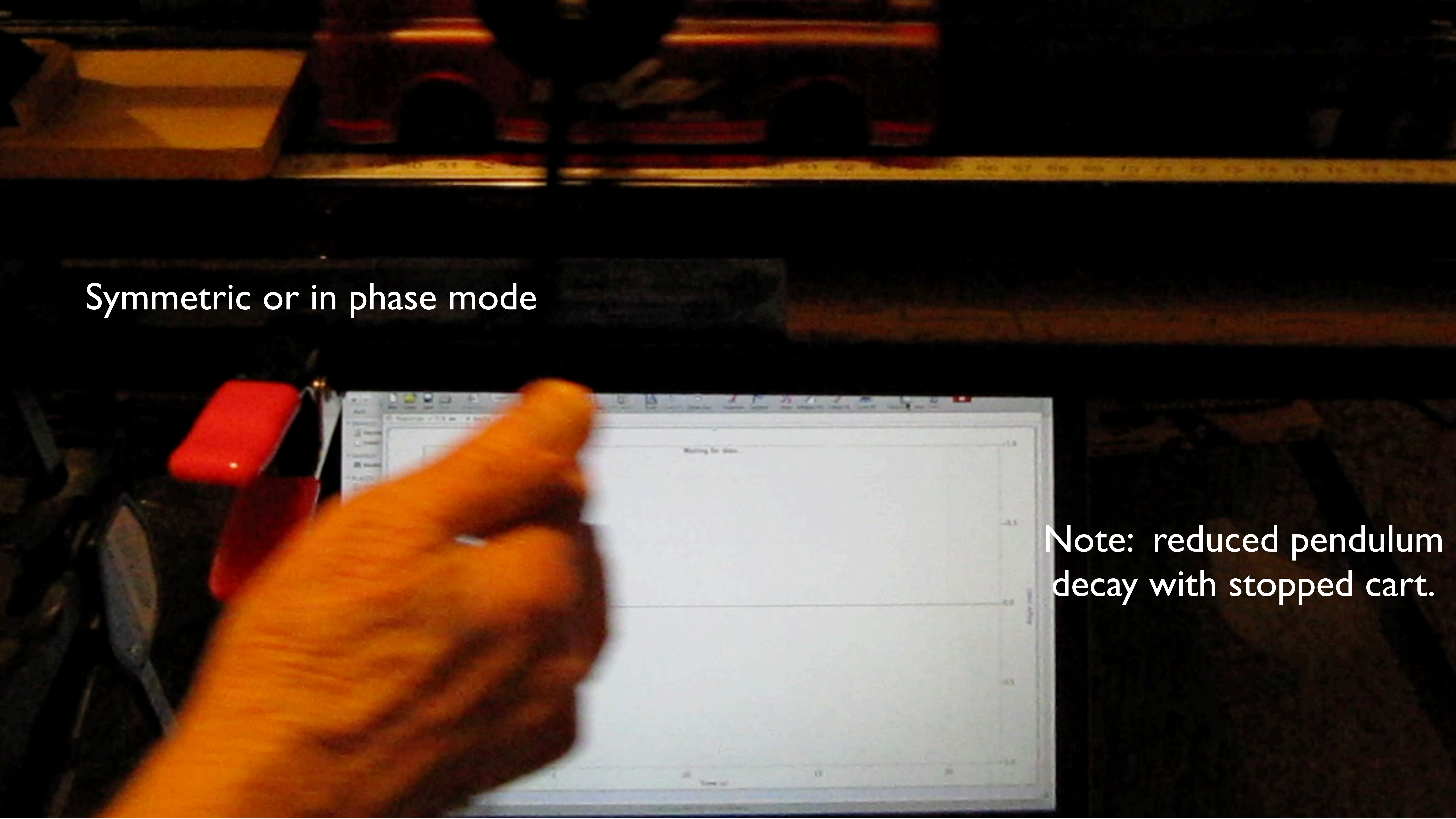
(without the cable, 39) Note the superior Coulomb fit.



Finally, some video clips of the oscillators “in action”.

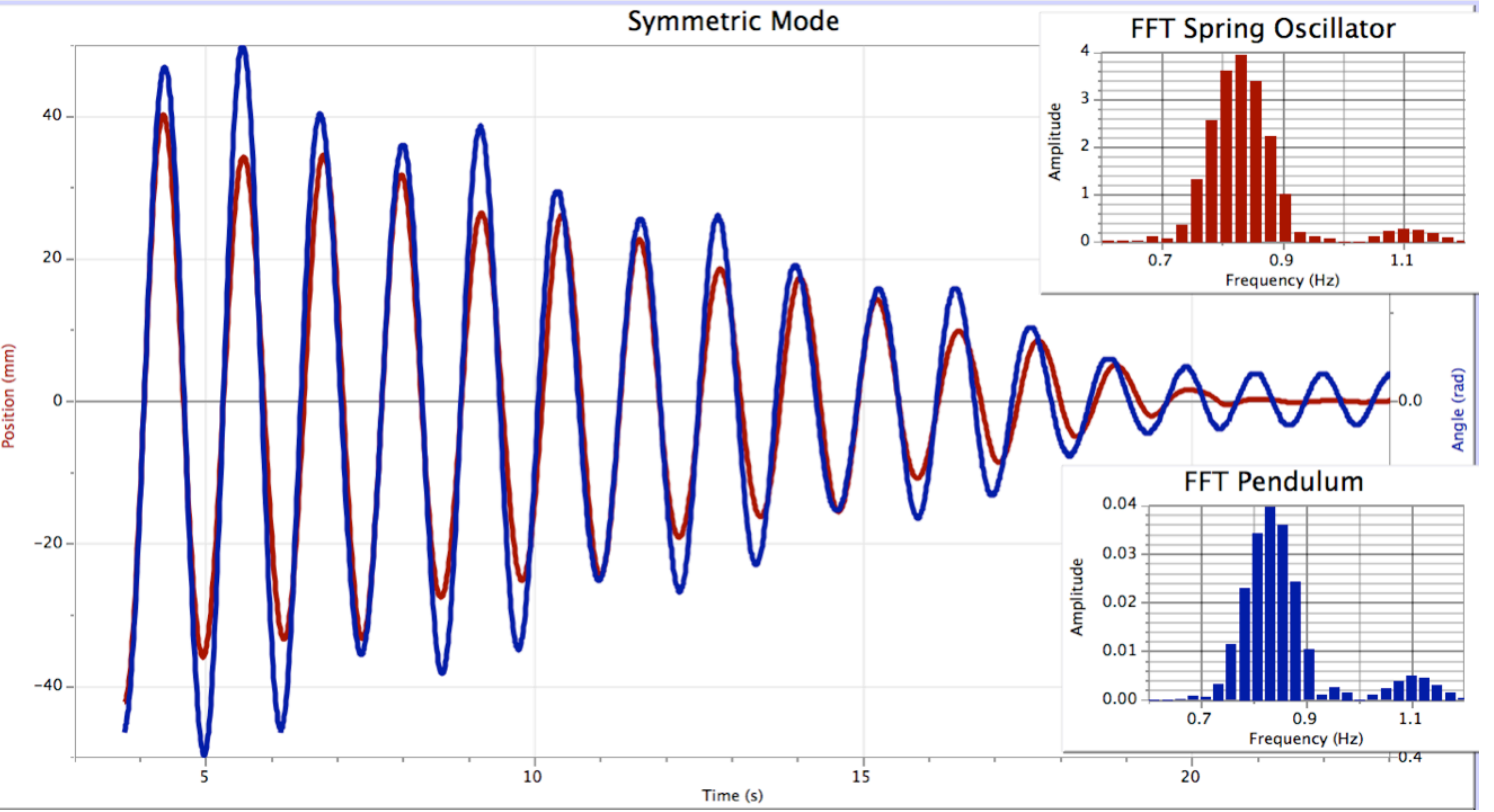
Two Degrees of Freedom, therefore, Two Normal Modes

The Symmetric and Antisymmetric video clips followed by their FFT analyses



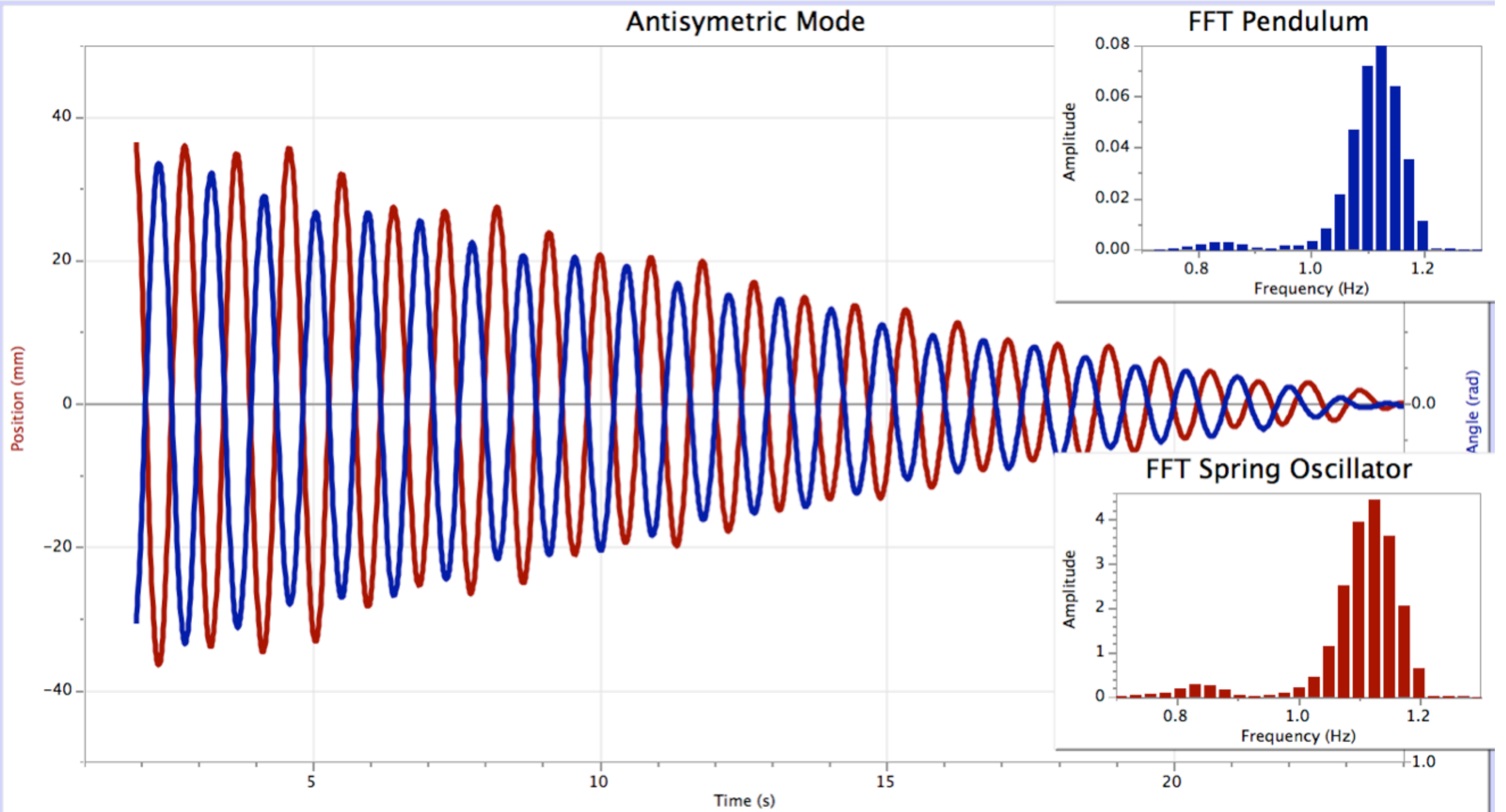
Symmetric or in phase mode

Note: reduced pendulum decay with stopped cart.

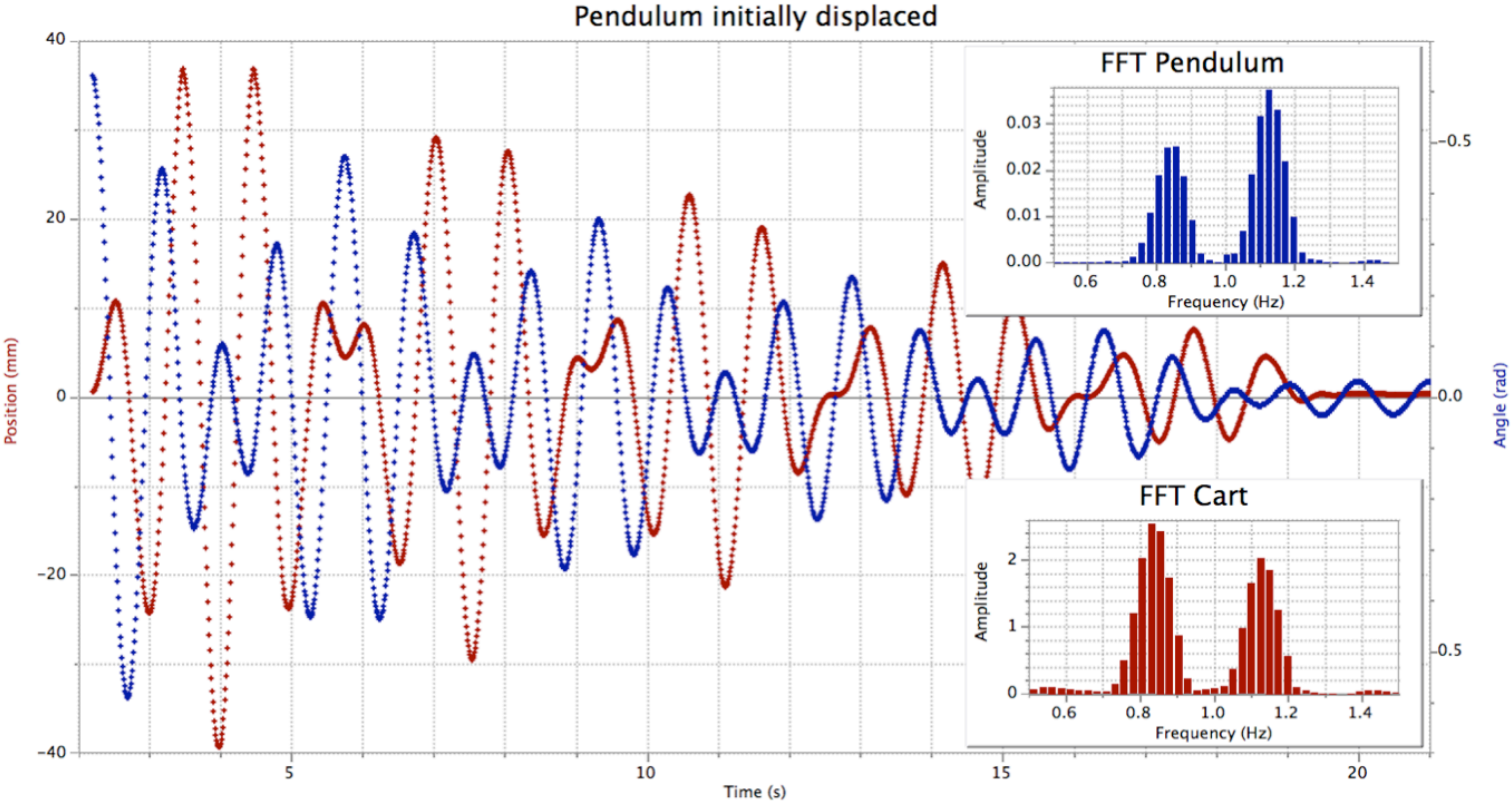


Symmetric Mode
FFT Spring Oscillator
Amplitude
Frequency (Hz)
FFT Pendulum
Amplitude
Frequency (Hz)
Position (mm)
Angle (rad)
Time (s)

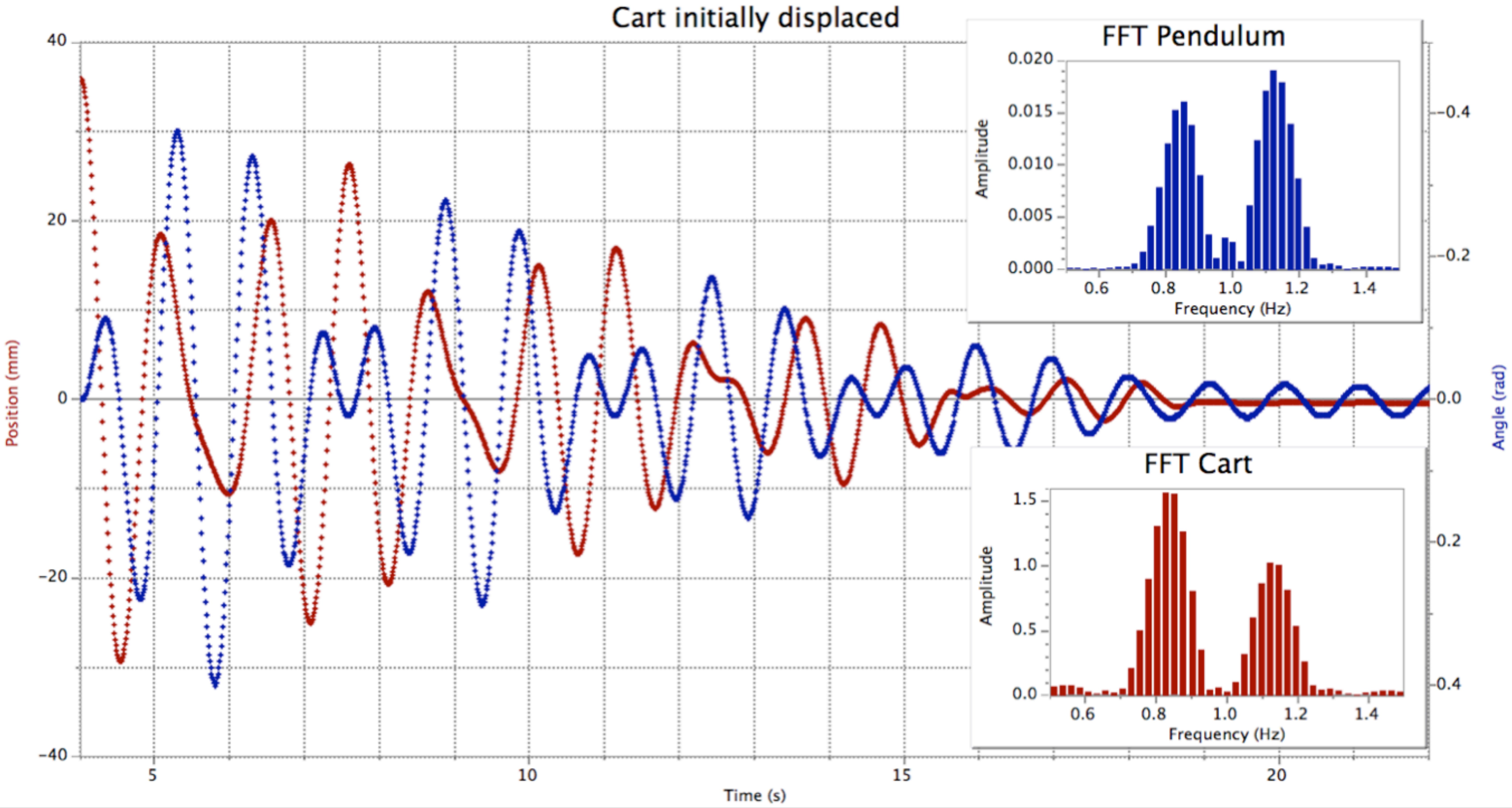
Antisymmetric mode


Antisymetric Mode
FFT Pendulum
FFT Spring Oscillator
Position (mm)
Angle (rad)
Time (s)
Amplitude
Frequency (Hz)
40
20
0
-20
-40
5
10
15
20
0.0
1.0
0.08
0.06
0.04
0.02
0.00
0.8
1.0
1.2
4
3
2
1
0

The following two graphs show the cart initially held in its equilibrium position until the pendulum, displaced, is released, and the cart displaced.



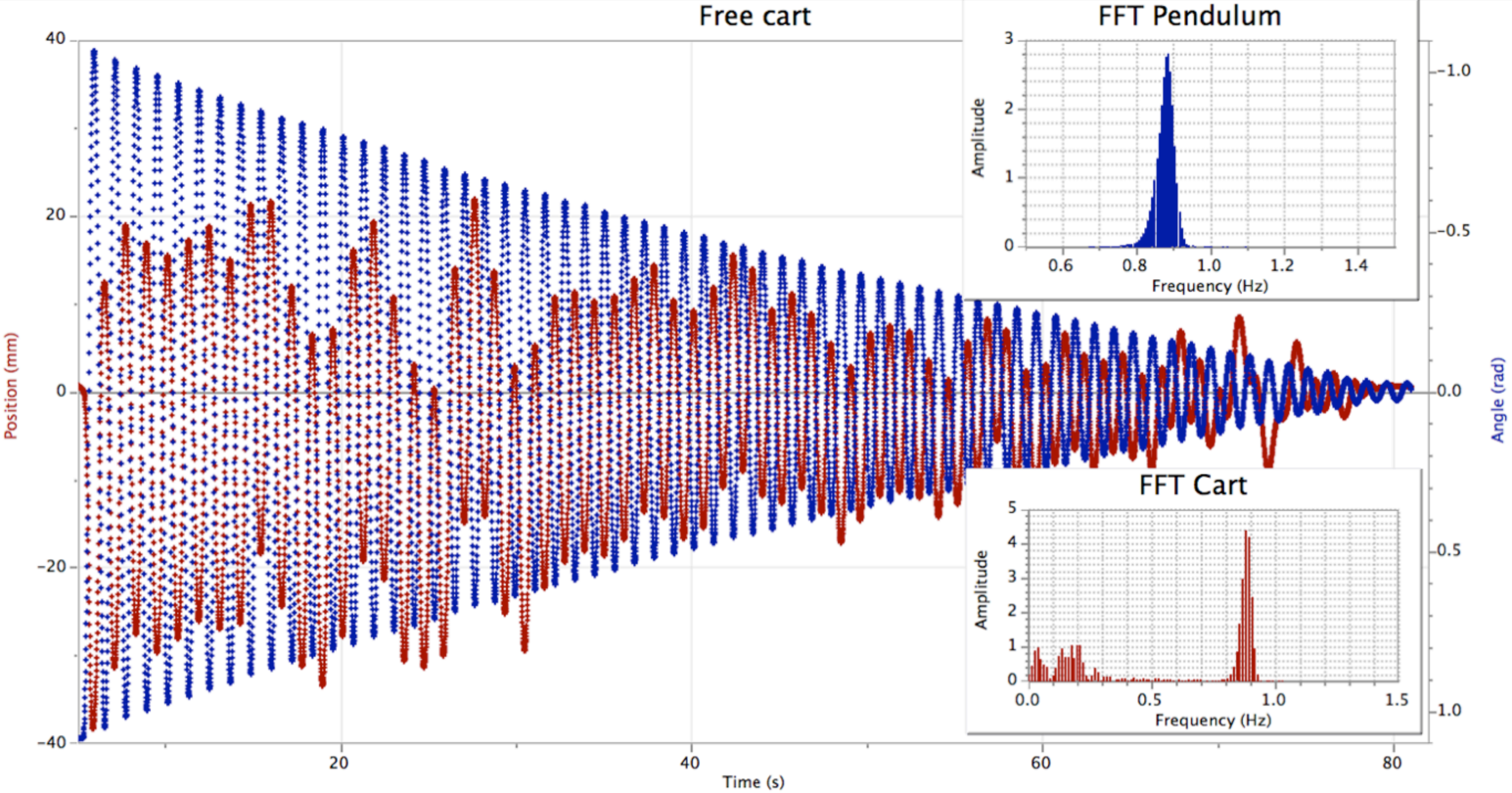
Pendulum initially displaced
Position (mm)
Angle (rad)
Time (s)
40
20
0
-20
-40
5
10
15
20
-0.5
0.0
0.5
FFT Pendulum
Amplitude
Frequency (Hz)
0.03
0.02
0.01
0.00
0.6
0.8
1.0
1.2
1.4
FFT Cart
Amplitude
Frequency (Hz)
2
1
0
0.6
0.8
1.0
1.2
1.4



Cart initially displaced
FFT Pendulum
FFT Cart
Amplitude
Frequency (Hz)
Position (mm)
Angle (rad)
Time (s)
40
20
0
-20
-40
5
10
15
20
-0.4
-0.2
0.0
0.2
0.4
0.000
0.005
0.010
0.015
0.020
0.0
0.5
1.0
1.5
0.6
0.8
1.0
1.2
1.4

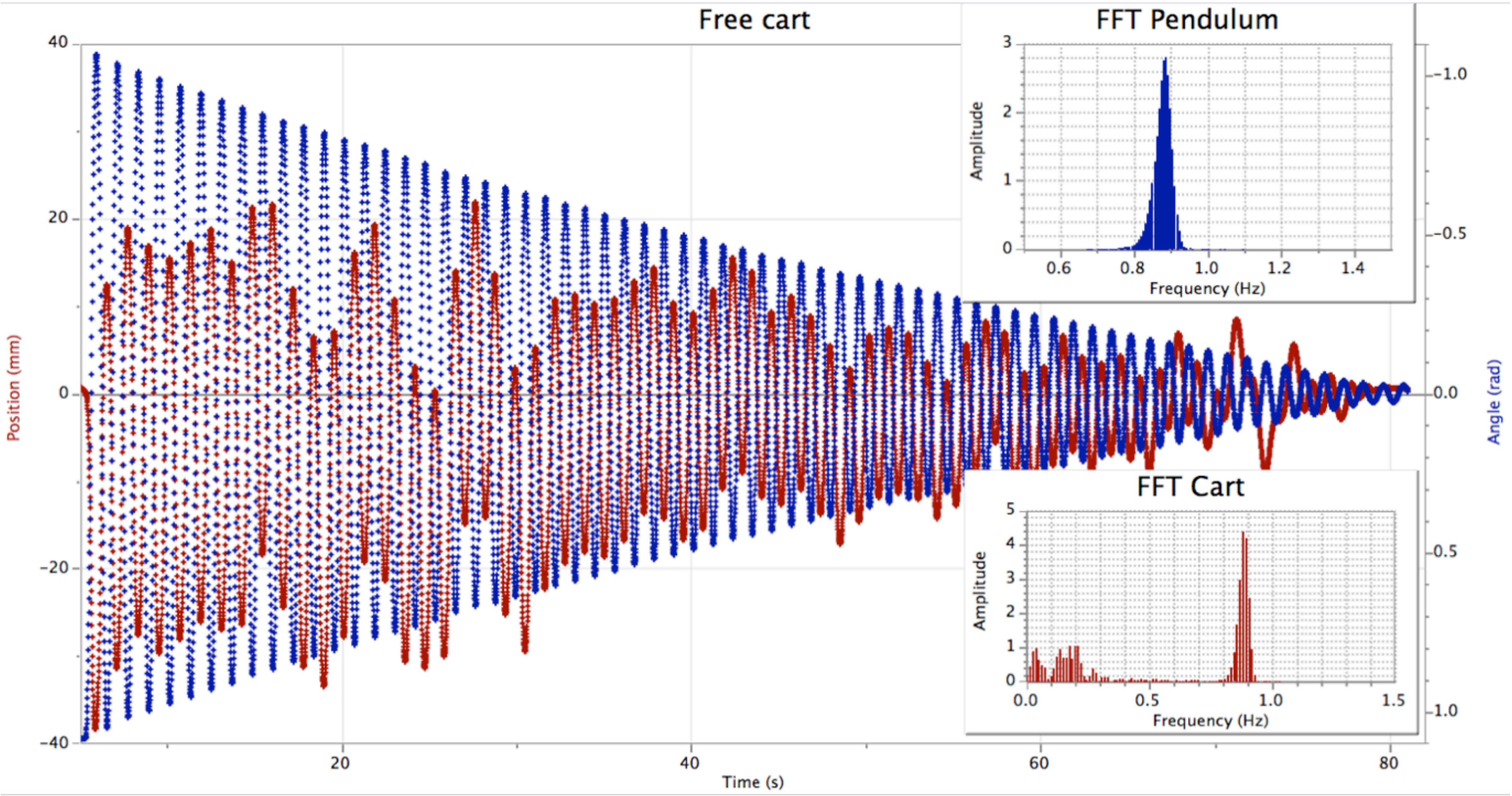
And finally, a video clip of the system's behavior with a free, no springs, cart followed by the graph.

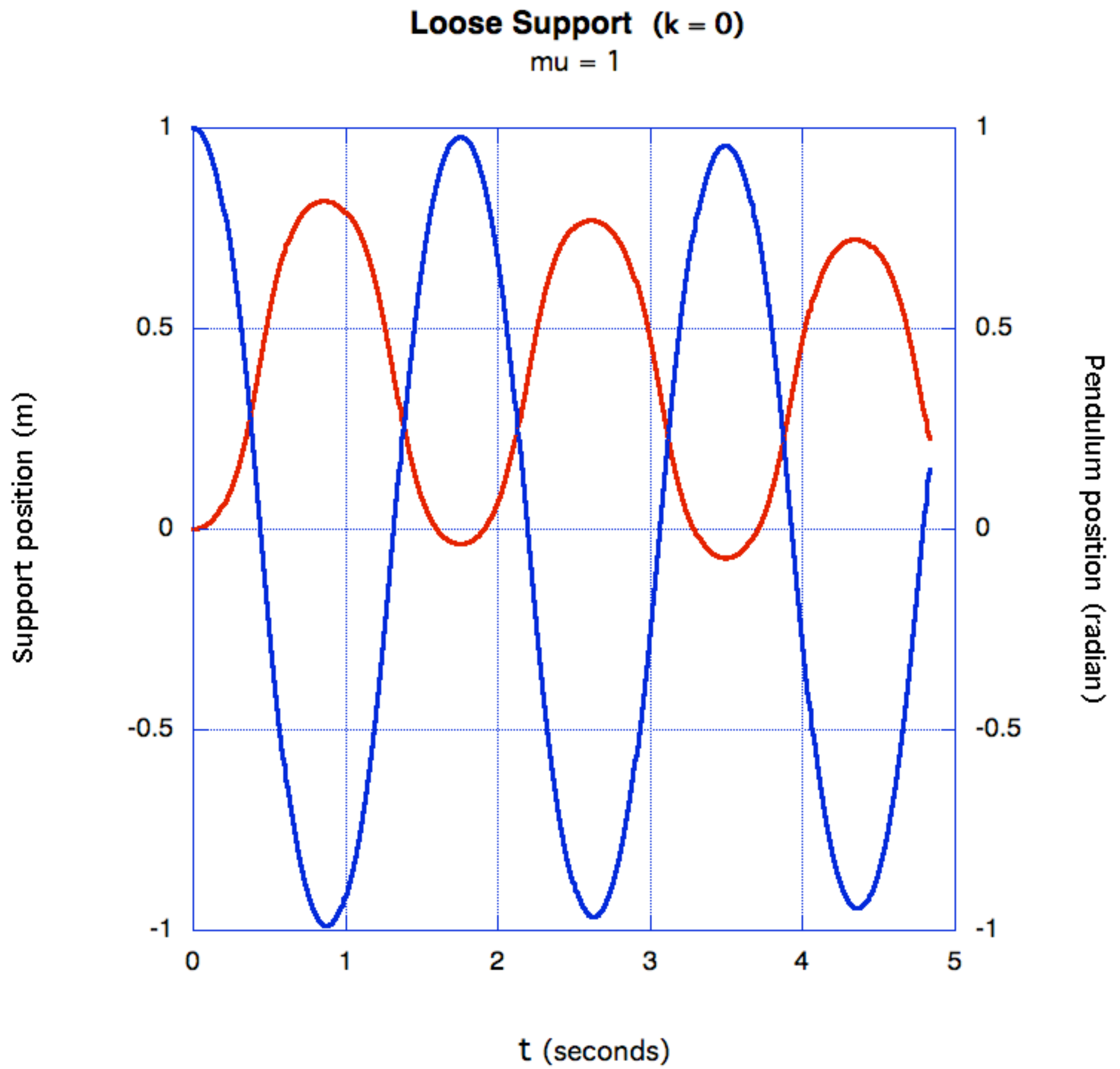

ME-6950



Free cart
Position (mm)
Angle (rad)
Time (s)
FFT Pendulum
Amplitude
Frequency (Hz)
FFT Cart
Amplitude
Frequency (Hz)

Note the difference between reality and numerical modeling! The apparatus may even be showing chaos.



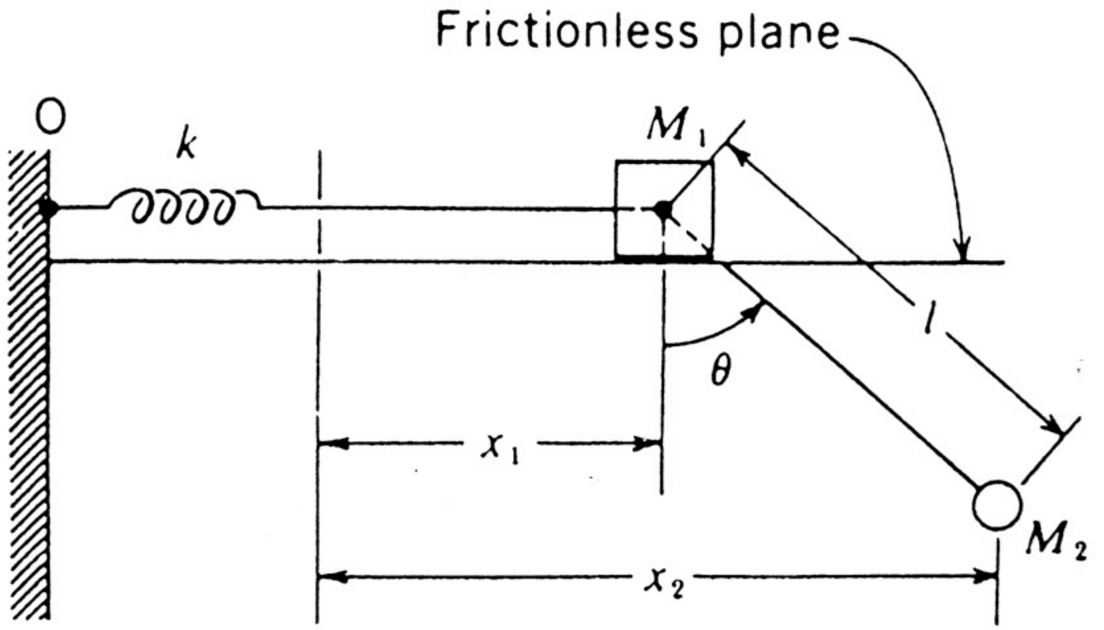


A number of modern undergraduate texts analyze coupled oscillators including A. P. French's introductory text. He asks as an exercise*: Solve for the characteristic (eigen) frequencies and amplitudes of the spring and pendulum coupled oscillator system with equal masses and the pendulum linearized. I have solved the secular determinant for unequal masses to find the normal frequencies.

*Vibrations and Waves, p. 156 problem 5-11

Here's that problem.

5–11 The sketch shows a mass M_1 on a frictionless plane connected to support O by a spring of stiffness k. Mass M_2 is supported by a string of length l from M_1.



(a) Using the approximation of small oscillations,

$$\sin\theta \approx \tan\theta \approx \frac{x_2 - x_1}{l}$$

and starting from $F = ma$, derive the equations of motion of M_1 and M_2:

$$M_1\ddot{x}_1 = -kx_1 + M_2\frac{g}{l}(x_2 - x_1)$$

$$M_2\ddot{x}_2 = -\frac{M_2 g}{l}(x_2 - x_1)$$

And the solution:

By dividing by the masses appropriately, and letting $k/M_1 = \omega_s^2$, $g/l = \omega_p^2$, $\omega_\$^2 = \Omega_\$$,

and $\mu = M_2/M_1$, one obtains: $\ddot{x}_1 + (\Omega_s + \mu\Omega_p)x_1 - \mu\Omega_p x_2 = 0$, and

$$\ddot{x}_2 + \Omega_p(x_2 - x_1) = 0$$

Subbstituting the trial solutions $x_\$ = B_\$ e^{i\omega t}$, collecting, cancelling the common factors, and setting the determinant of the $B_\$$ coefficients to zero, one obtains:

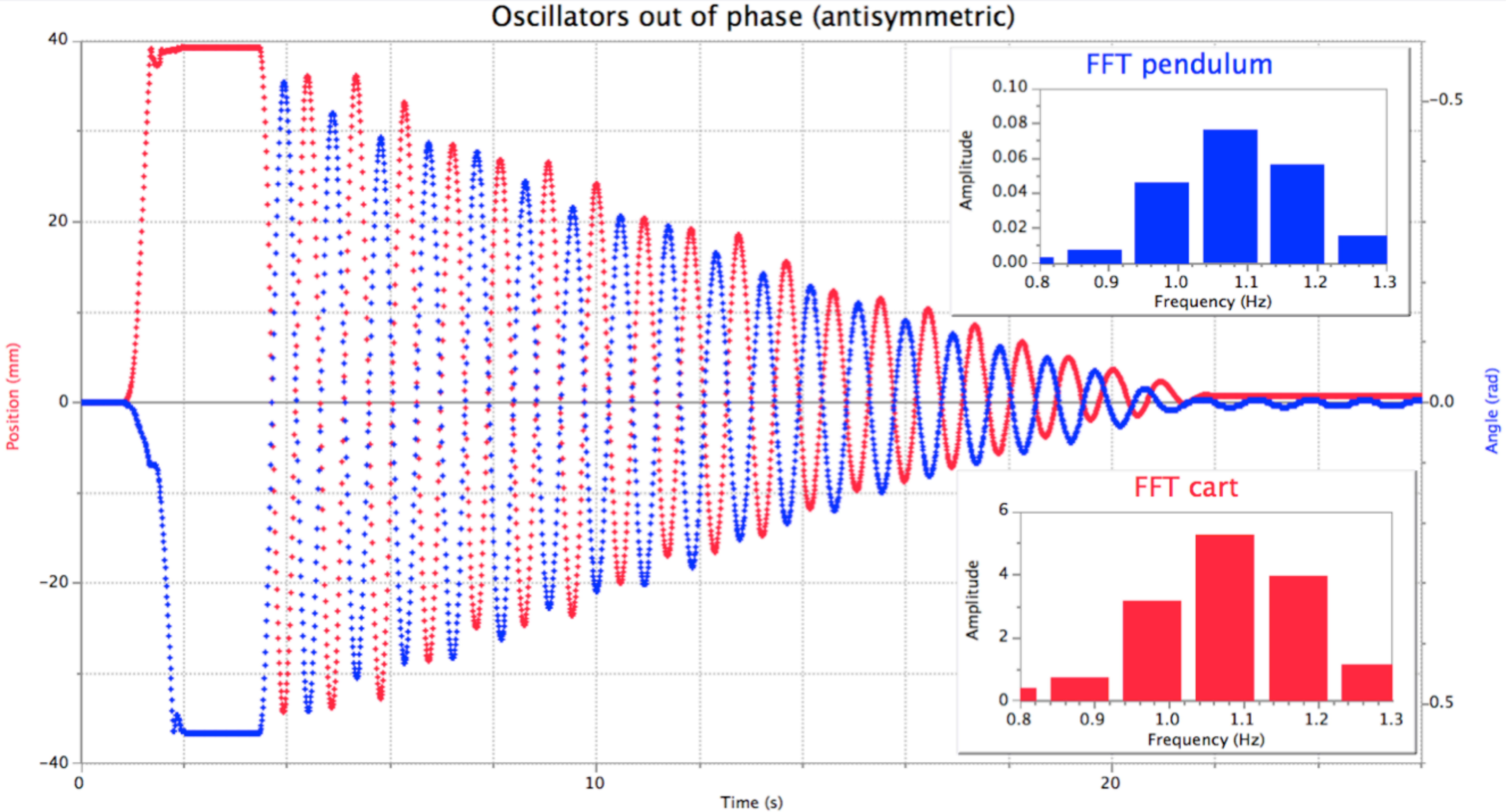
$$\Omega^2 - (\Omega_s + (1+\mu)\Omega_p)\Omega + \Omega_s\Omega_p = 0$$

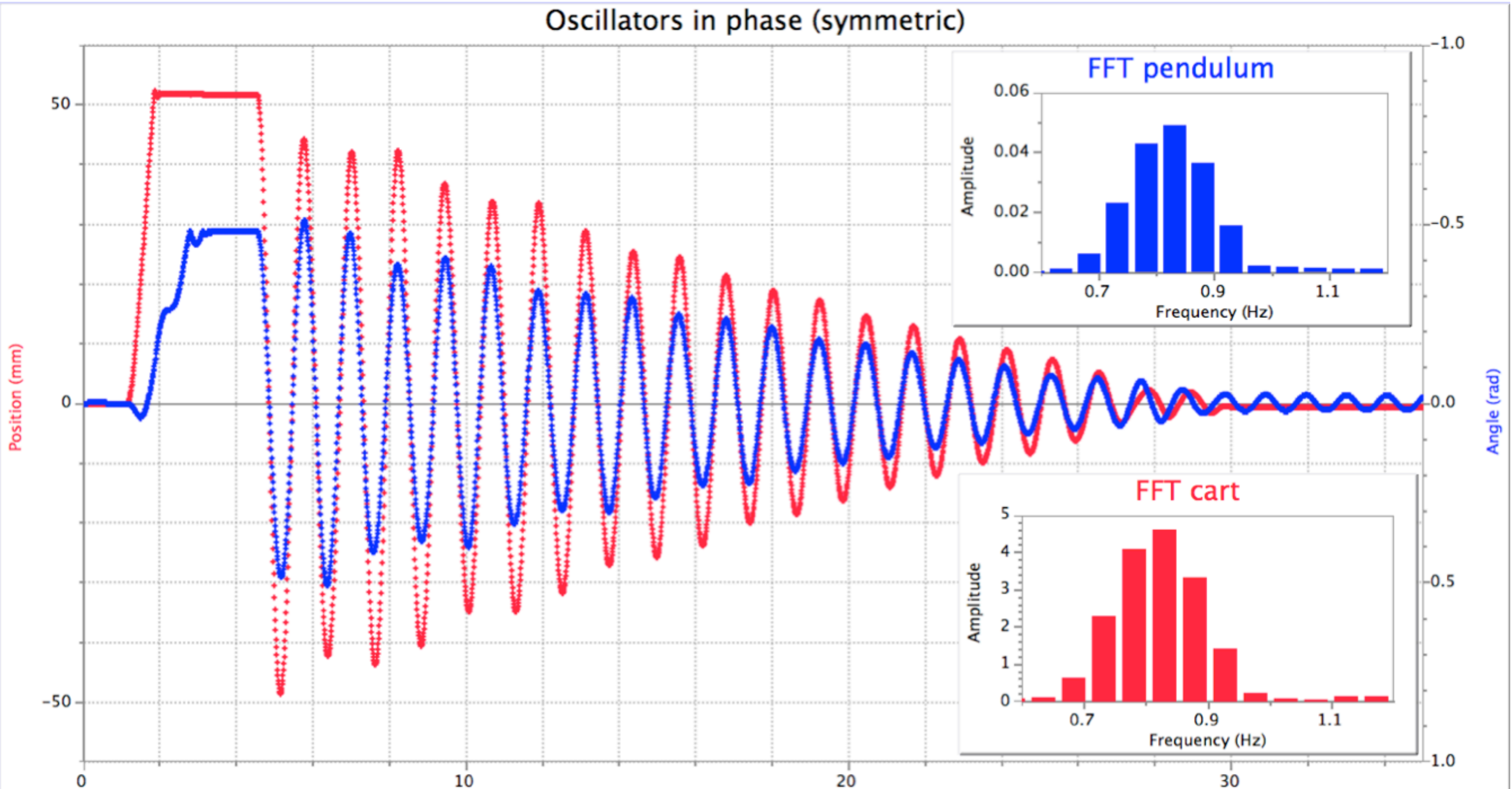
Substituting into the quadratic formula, and "undoing" the Ω substitutions, one obtains

$$\omega = \sqrt{\frac{(\omega_s^2 + (1+\mu)\omega_p^2)}{2} \pm \sqrt{\frac{(\omega_s^2 + (1+\mu)\omega_p^2)^2 - 4\omega_s^2\omega_p^2}{4}}}$$

Remember: The above is derived from a linearized pendulum.

The following two graphs are of the oscillators' two normal modes with their constituent frequencies set nearly equally. The previous ones differed by approximately six percent.


Oscillators out of phase (antisymmetric)
FFT pendulum
Amplitude
0.00
0.02
0.04
0.06
0.08
0.10
0.8
0.9
1.0
1.1
1.2
1.3
Frequency (Hz)
FFT cart
Amplitude
0
2
4
6
0.8
0.9
1.0
1.1
1.2
1.3
Frequency (Hz)
Position (mm)
40
20
0
-20
-40
Angle (rad)
-0.5
0.0
-0.5
0
10
20
Time (s)



Oscillators in phase (symmetric)
FFT pendulum
Amplitude
Frequency (Hz)
FFT cart
Position (mm)
Angle (rad)

Both oscillators' isolated (natural) frequencies are approximately 5.9 radian/s.
The pendulum and cart masses are 0.15 and 1.92 kg.

Mode	Experimental	Analytic
Symmetric	5.2 radian/s	5.1
Antisymmetric	7.2 radian/s	6.8

Attempted answers and copies:

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