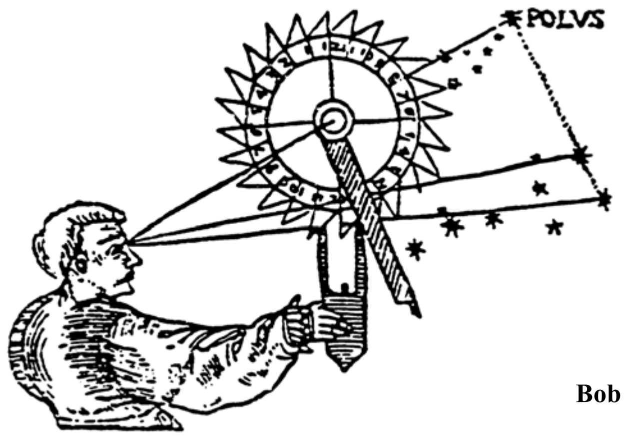

HOROLOGICAL SCIENCE NEWSLETTER NAWCC CHAPTER #161

www.hsn161.com

Issue 2016 - 1 ***January 2016***

Bob Holmström, Editor & Publisher, 2934 NW 53rd Drive, Portland, OR 97210
Phone: 503-292-3685, holmstro@gmail.com
David Spong, Secretary & Treasurer, 31120 Marne Drive, Ranch Palos Verdes, CA 90275 Phone: 310 541-4150, edavid.spong@verizon.net

<u>Page</u> <u>Contents</u>

Material is needed for the next issue – there are no articles in the queue.

<u>**The deadline for material for the next issue is March 15th**</u>

Bob Holmström

Above: Drawing from Apians cosmographicus Liber of 1533, showing how a nocturnal is used to tell the time at night from the Great Bear.

Measuring the Simple Harmonic Motion of a Pendulum

Bryan Mumford and Douglas S. Drumheller

1 Introduction

Ever since its invention by the mathematician Christiaan Huygens in 1656, the pendulum clock has been one of the most studied mechanisms in science and engineering. It's not often appreciated that most of what we know about the pendulum is based on the notion that it undergoes simple harmonic motion (SHM). For example the familiar relationship between the length of the rod L, the acceleration of gravity g and the beat period T,

$$T = \pi\sqrt{L/g}, \tag{1}$$

depends on the pendulum following SHM. Of course circular error causes the actual value of the beat to vary slightly from this prediction, and indeed it is also known that this variation in beat is associated with a tiny deviation of the motion from SHM. For a pendulum swinging in a semi arc of a degree or two the predicted deviation from SHM due to circular error is about 1 part in ten thousand. (See Eq. 6 of Drumheller, Journal of Applied Mechanics, 2012, Vol. 79.)

Now SHM is a mathematical concept. It means the angle of arc of motion θ of a pendulum can be represented by a function such as

$$\theta = A \sin \pi t/T, \tag{2}$$

where t is the time and A is the amplitude or semi arc of the motion. We notice that during SHM the amplitude is always A for all past and future times. Thus a clock that is allowed to coast down by disengaging its escapement does not exhibit SHM because the damping of the motion causes the amplitude to continually decay with time. Moreover we have an easy way to estimate the decrement in amplitude ΔA during every two beats or one complete oscillation. It is

$$\frac{\Delta A}{A} = \frac{\pi}{Q}. \tag{3}$$

(See Drumheller, HSN 2014-4, pp. 6-12.) Here Q is the much maligned "quality" parameter, which represents the ratio of the total energy and the energy lost per swing. A typical value of Q for a long-case clock is about $2000 \times \pi$, and thus we see that the aerodynamic damping of the pendulum causes the amplitude to decay about 0.0005 parts of the current amplitude. Thus as the amplitude drops so does the decrement and in theory the pendulum never stops swinging.

Now if you think about it a typical escapement acts on the pendulum only for a fraction of the total swing. During the time that it does not act, the amplitude of the swing is decaying. The escapement is designed and carefully adjusted to restore the amplitude. Just as the free swinging decay represents a departure from SHM the action of the escapement is aimed at restoring SHM.

You might argue that some escapements produce large impacts that momentarily cause the motion to wildly deviate from SHM. We would argue that while its possible to design such mechanisms, all have been discarded because they produce bad clocks. We argue that accuracy of a clock depends upon the motion remaining close to SHM.

We actually started this project with a practical objective in mind—to clarify an algorithm used in the MicroSet timer. In 2004 a new feature was added to it. This was the "amplitude" parameter which allowed you to measure and plot A. It works by reading the duration of time that the optical sensor is blocked at the center of swing, often called bottom dead center (BDC).

Now as this is a direct measure of the velocity of the bob, how does it give you an indication of the amplitude of the bob? The answer is found in the assumption of SHM. It is a simple mathematical operation to compute the velocity from Eq. 2. It tells us that the amplitude of the velocity V of the bob is related to the amplitude of the motion A by the expression

$$A = VT/\pi. \tag{4}$$

But it tells us even more. We find that the profile of the velocity of the bob is a perfect copy of the motion that is scaled by the fundamental frequency, π/T which is expressed in radians per second It also tells us this profile is time shifted by half a beat, $-T/2$, so that the point of maximum velocity occurs exactly at BDC.

As MicroSet measures V and T it can compute the value A from this simple relationship. Thus the existence of SHM allows an easy and reliable short cut to determining A. The alternative is to measure the amplitude with your eye. This is exceptionally imprecise unless you go to the extreme, as someone did, by using a single hair plucked from the back of his faithful (white) dog, fastened vertically to the moving pendulum, illuminated with a visible laser beam and then observed from across the room with a telescope for precision.

Our short cut, of course, depends upon the bob actually following SHM. There is a large horological contingent that is often suspicious of a mathematical concept and might question the relevance of SHM to the accuracy of a pendulum clock–indeed to any clock. The obvious answer to their objection is to measure the motion of a pendulum and see just how much it actually deviates from SHM, and that's what we have done. So as to not keep you waiting we found that the deviations are quite small–about one part in a thousand and perhaps less.

2 The SHM Experiment

Our goal is to measure the deviation of the actual motion of a pendulum from SHM. It's not an easy task. But in 2012 circumstances presented us with the key—a Renishaw linear encoder donated by horological experimenter and CNC professional Mike Everman. This encoder uses an optical read head placed close to a precision encoder strip. As the read head moves over the striped strip, it outputs a signal transition at every edge. The timing of these transitions indicates the motion of the read head to a resolution of one micron. (Actually two signals are produced by two read heads oriented in quadrature so as to detect motion reversals.)

The clock we have used is a Self Winding Clock Company Model 41 one second pendulum with a dead-beat escapement and a style "F" reciprocating rewind. A detailed description of the apparatus, data acquisition and signal processing can be found in the Appendix. After installation of the apparatus this clock was going for several days before the data were acquired.

3 Deviation of the Motion from SHM

As described in the Appendix a record of 100 beats of the pendulum was obtained. This record gives the position of the bob in meters of arc length at uniformly spaced time intervals. A portion of this record is shown in Fig. 1. You should recall that we are measuring the motion of the encoder strip and

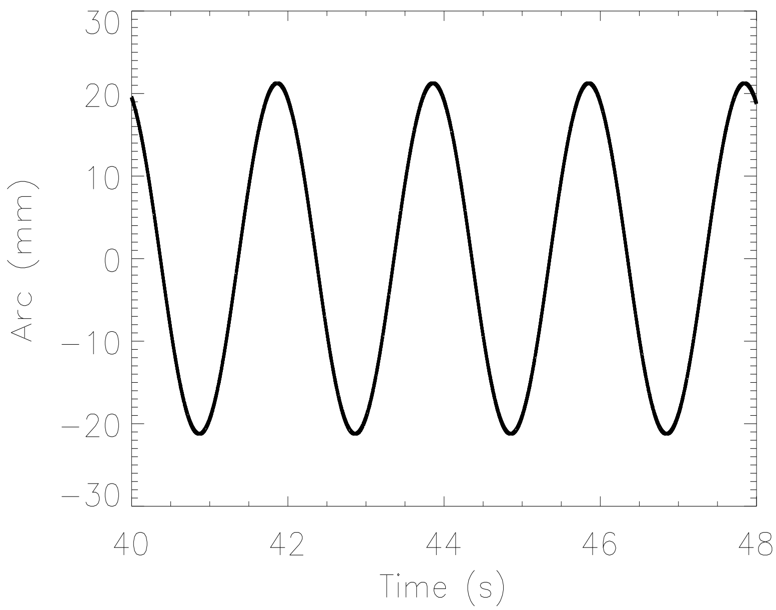

Figure 1: The motion of the pendulum rod.

not the motion of the bob. Thus the amplitude of the oscillation is about 22 mm to each side of bottom dead center (BDC). This corresponds to a total arc of 3.05 degrees. The added damping produced by the apparatus has reduced the arc from the original value of 4.35 degrees. We will also see that the added mass of the apparatus has caused the rate to increase from 0.5 Hz to approximately 0.5058 Hz.

To a casual observer the shape of this motion curve appears to be a sine wave, but it surely must deviate from this form. If it actually represents SHM it can be represented by one sine wave with one frequency, the beat frequency of the clock. But if its not SHM then other sine waves of different frequencies will be present. To find out just how many different frequencies are present let's take the Fourier transform of this data. The results of that operation are shown in Fig. 2. If we were to simply plot the amplitudes of all the frequencies on a linear vertical scale we would only see one frequency component at about 0.5058 Hz[1]; however, instead we have done what is normally done and not

[1] The fast Fourier transform actually only evaluates the spectral amplitudes at relatively

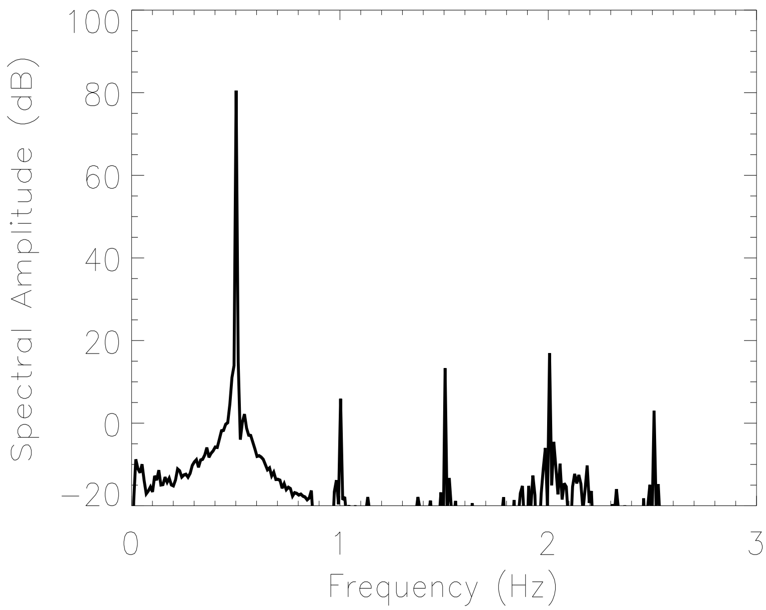

Figure 2: The frequency content of the motion.

plotted the amplitude but rather 20 times the logarithm of the amplitude. This allows us to see details of the much weaker peaks. The resulting unit is called a dB or decibel. The large fundamental peak is at 80 dB and the others, called the harmonics, reside just below 20 dB. If the fundamental at 0.5058 Hz were the only peak to appear in this spectrum, we would conclude that the motion is SHM. Ideally the presence of the harmonics represent deviation of the motion from SHM, and practically they may also represent error in the alignment of the encoder strip apparatus.

You should realize that when plotted on this scale peaks that are 60 dB lower than the fundamental represent harmonics with amplitudes that are only a thousandth of the fundamental and energy levels that are a millionth of the fundamental. There are also harmonics above 3 Hz, but their peaks are below 0 dB corresponding to amplitudes that are less than a ten-thousandth of the fundamental.

To see exactly where the actual motion deviates from SHM we now remove

widely spaced increments of frequency. The most accurate estimate of the fundamental frequency can be obtained from either a digital Fourier transform routine or an auto correlation routine as discussed in Section 4.

the fundamental component from this spectrum by zeroing it out between 0 and 0.9 Hz. Then we invert the spectrum back into the original time domain. Two curves are shown in Fig. 3. The dashed line is included as a reference as

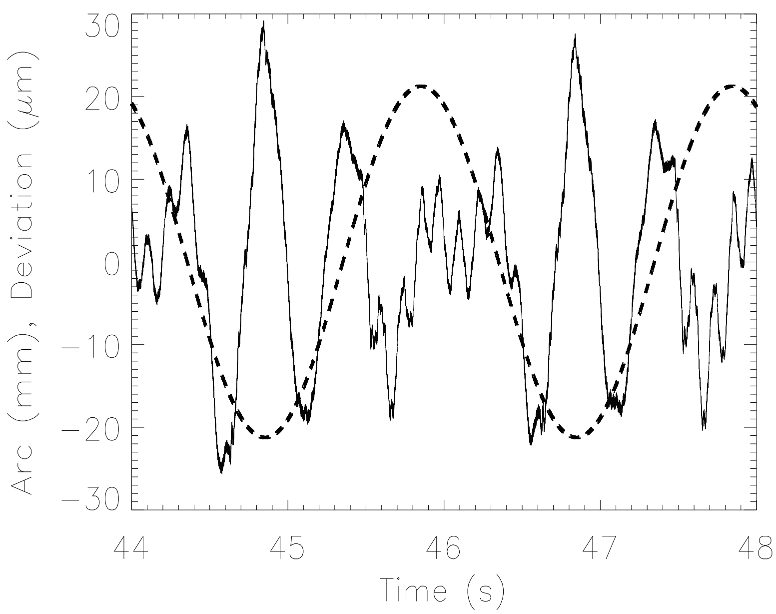

Figure 3: The deviation of the motion from SHM (solid line) compared to the total motion (dashed line).

it is the original measured motion. The vertical scale is read as millimeters of arc length. It is essentially the same plot as Fig. 1; albeit, plotted over a shorter time interval. The solid line is the motion after the fundamental has been removed. In this case the vertical scale is read as micrometers of arc length. Thus this curve has been magnified vertically by a factor of 1000. This is the deviation of the measurement from SHM.

Our estimates of the deviation from SHM suggest that decay of the 22 mm amplitude due to aerodynamic drag will cause deviations on the order of $.0005 \times 22 = 11$ μm, the circular error will cause deviations on the order of $0.0001 \times 22 = 2$ μm and the misalignment of the encoder strip will cause deviations on the order of $0.0007 \times 22 = 15$ μm. Thus the solid line in Fig. 1 is most likely a representation of real deviation corrupted by error due to the misalignment of the encoder strip and other factors such as out-of-plane motion of the pendulum or wobble about the axis of the pendulum rod. Moreover, as we have not directly measured the motion of the center of gravity of the bob, the bob might be undergoing nearly perfect SHM while the rod, trapped between the bob and the spring mount at the pivot could be vibrating like a violin string.

Alternatively the deviation curve in Fig. 3 could have been constructed by fitting a simple sine wave to the original data and then subtracting the result. We have chosen the Fourier transform method because it is less subjective and the spectrum reveals other important information. For example, suppose we number the harmonic peaks. The plot shows the fundamental, peak 1, and two odd harmonics, peaks 3 and 5 as well as two even harmonics, peaks 2 and 4. The even harmonics represent deviations that are asymmetrical about BDC. For example, if an even harmonic causes a negative deviation on the left beat it will cause a positive deviation on the right beat. This might occur if the encoder strip were tilted and not exactly square to the pendulum rod. The odd harmonics produce symmetrical deviations as might occur if the encoder strip were too high or too low. Previous analysis also shows that circular error causes deviation from SHM to appear in the third harmonic.

4 The Point of Maximum Velocity

One of the more intense discussions in horology can be found starting in the July 1989 issue of the Horological Journal. It begins with an article on Airy's equations by Philip Woodward (See Page 16.) and continues with a series of letters to the Editor.[2] The argument focuses on Airy's equations for computing escapement error. Airy's results depend on the angle between the action of the escapement and BDC, and it is argued that they should depend on the angle between the escapement and the point of maximum velocity. Woodward in particular counters that that is *Clutching at Straws* as the deviation of the point of maximum velocity from BDC is very small. We note with interest that no one in this discussion presents any experimental data. Here we have an opportunity to change that. As we have just detected tiny deviations of the motion from SHM, let's see if we can detect a deviation of the maximum velocity from BDC.

[2]See Pages 115, 186, 291, 292, 327, 328, 398 and 399 in HJ 1989 as well as Pages 3, 74 and 75 in HJ 1990.

To do this we must first determine the velocity of the pendulum by numerically differencing the data. Let the quantity $x(i)$ represent the i^{th} value of the motion in our list of data, which is measured at the time $t(i)$. Then the velocity $v(i)$ is[3]

$$v(i) = \frac{x(i) - x(i-1)}{t(i) - t(i-1)}. \tag{5}$$

Now all we need to do is plot these velocity values alongside the motion data to see if the maximum velocity occurs when the motion passes through zero. However, we notice that the amplitude of the velocity is not numerically equal to the amplitude of the motion, and thus these two curves just won't fit together very well on the same plot. So let's take a hint from Eq. 4 and scale all the velocity values by dividing them by the fundamental frequency in radians per second, the ratio π/T, to see if they then fit on the same plot. The motion (solid line) and the resulting scaled velocity (dashed line) are shown in Fig. 4. This plot suggests that the maximum of the velocity occurs at BDC, but its still a little difficult to tell if there's some slight error in timing. So let's try shifting the velocity curve to the right by exactly one half a beat to see if it will overlay the motion curve. To accurately determine the beat time from the motion data alone we use a signal analysis process called an auto correlation. This process matches a copy of the motion data, which is time shifted two beats, against the original data to determine the exact time shift required to exactly overlay the shifted copy onto the original. In this case it's equivalent to averaging 100 MicroSet measurements to determine the beat. From this we determine the beat is 0.986582 s. Thus the fundamental frequency is 0.5068 Hz. So now we can shift the velocity data to right by half a beat and plot it again. We've already done that. Look very closely again at the solid line. It's really two lines–the original motion curve and the overlayed time-shift velocity curve. Clearly the velocity is out of phase from the motion by half a beat, which means the maximum of the velocity occurs at BDC or at least it does to within the accuracy of our plotting process. Indeed as predicted by the assumption of SHM we now see that the velocity curve is a scaled copy of the motion that is time shift half a beat.

[3]There's a little hitch here. As the pendulum moves slowly past the maximum amplitudes of swing, the encoder often does not encounter a strip edge as the time advances from one sample to the next. Thus the measured motion has a staircase structure; albeit, a very fine one as the steps are only a micrometer high. To obtain a realistic representation of the velocity we first run the motion data through a boxcar filter and average each of the one million motion samples with 49 of its nearest neighbors.

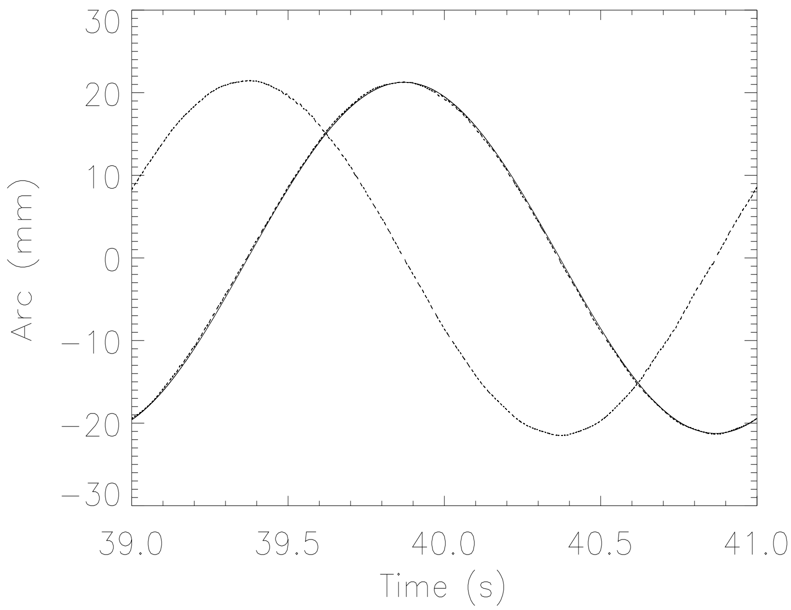

Figure 4: Comparisons of the motion (solid line) and the velocity (dashed line).

However, there is an even more accurate way of determining the time shift between the motion and the velocity. It's called a cross correlation. When this process is applied to the data the best time shift is slightly less then half a beat, 395 μs less, which implies that maximum velocity occurs slightly after BDC. This time difference corresponds to a distance of about 27 μm or 0.001 in. Indeed that's one-third the thickness of a sheet of paper and a very small change in the angle appearing in Airy's equations.[4] We suspect this measurement is affected by the resolution capability of our apparatus, but may still represent a real deviation of the maximum velocity from BDC.

[4]As noted, Woodward claims worry over the distance between BDC and the point of maximum velocity is *Clutching at Straws*. Is it? Well we are unaware of the dimensions of English straw; however, common New Mexico straw is a hollow tube with a diameter of approximately 0.125 in and a wall thickness of 0.010 in. Thus its dimensions are huge in comparison. Surprisingly, even the dog-hair apparatus described in our introduction lacks the precision to resolve this tiny deviation because, as is often true, the hair of the dog (about 0.003-in diameter) is just too thick.

5 Conclusions

We have measured the motion of a pendulum with a dead-beat escapement using a linear encoder. The results show the deviation of the motion from simple harmonic to about one part in 1000 or less. The maximum velocity is predicted by Eq. 4 and occurs slightly beyond BDC. Possibly half of the deviation we have measured is an artifact of the misalignment of the apparatus. For future tests the encoder strip should be mounted directly to the center of gravity of the bob with a better control of the alignment by perhaps using a dial indicator. It may even be possible to use the harmonic components in a frequency spectrum such as Fig 2 to align the strip.

Philip Woodward summed up his arguments in Horological Journal by quoting the Nobel Physicist Richard Fenyman—the real success comes from *knowing what is big and what is small in a given complicated situation.* We hope our data lends help in knowing what is small.

6 Acknowledgments

We wish to thank Bob Holmstrom and Tom Van Baak for reviewing the manuscript and suggesting changes. Also Bob using his excellent library found the HJ thread of arguments surrounding the point of maximum velocity, and Tom offered the violin analogy to describe one of the harmonic distortions that has been of concern to us.

A Appendix

Figure. 5. is a photograph of the experimental apparatus. The vertical rod is part of the pendulum of the clock. Under normal operation without the experimental apparatus the amplitude of swing of the clock bob is about 1.63 inches (41,400 microns) total arc, or 4.348 degrees.

Because the pendulum rod moves in an arc a Plexiglas fixture was fabricated and the encoder strip was mounted face down on its curved surface. The encoder head is then fixed to the clock case with an aluminum fixture arranged to hold the read head upside down. This support assembly is composed first of an aluminum bar bolted to the sides of the clock case. Next a section of aluminum t-bar is mounted on spring-loaded machine screws. The Renishaw read head is attached to the t-bar. This assembly is used to adjust

Figure 5: The Experimental Apparatus

the height and tilt of the read head, which then remains in a fixed position as the encoder strip moves above it.

The radius of curvature of the encoder strip mount is 21 inches. It was machined using a pivot rod on a milling machine and then mounted to the pendulum rod with a grooved Delrin rod and cable ties. This allowed us to slide the assembly up and down the rod until the encoder strip was exactly 21 inches below the pendulum pivot. As the pendulum swings, the ruled encoder strip swings past the read head at a nominally constant distance of 0.030 inches. However, the alignment was by eye so we expect that up to 0.015 inches of misalignment might be present. As the displacement of the encoder strip d is given by

$$d = R\theta, \tag{6}$$

where the radius of the strip is R, the error in the measurement of d is estimated to be

$$\frac{\Delta d}{d} = \frac{\Delta R}{R}. \tag{7}$$

Thus an error of

$$\frac{\Delta d}{d} = \frac{0.015}{21} = 0.0007 \tag{8}$$

might be present in the measurements and indistinguishable from a real deviation from SHM.

Because of the fast output rate of the quadrature encoder the digital signals were recorded as left and right audio inputs to an Alesis IO—26 music recording interface, which captures the bit stream as an audio file at a sample rate of 192,000 Hz. Each byte represents the state of the encoder at that instant, be it 1 or 0 depending on whether each read head sees a strip or just the gap between neighboring strips. Thus as long as the pendulum arc remains under 6 degrees we can capture its motion to one micron resolution and store the results as an 8-bit sound file on a computer. This WAV audio file is then converted to an ASCII text file. You can hear a sample of the raw audio data at the following URL:

http://bmumford.com/renishaw/renishaw.wav
(Warning: turn the volume of your speakers down first!)

This apparatus was used to capture the motion during 100 beats of the pendulum. Prior to the start of the experiment the clock had being going for several days. The resulting ASCII file was enormous and contained approximately 20 million motion samples in each recording channel. The record for each quadrature head was read at full resolution to detect motion reversals and reconstruct the history of the motion of the rod. We then determined that this record could be safely resampled at a much slower rate without aliasing the data. The final data set contains 1,000,000 samples of the motion over a time window of exactly 100 beats. It is important to window the data to an even integer number of beats so as to facilitate the most accurate digital Fourier transform.